

Stochastic Capacity Investment and Flexible versus Dedicated Technology Choice in Imperfect Capital Markets

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Abstract

This paper analyzes the impact of endogenous credit terms under capital market imperfections in a capacity investment setting. We model a monopolist firm that decides on its technology choice (flexible versus dedicated) and capacity level under demand uncertainty. Differing from the majority of the stochastic capacity investment literature, we assume that the firm is budget-constrained and can relax its budget constraint by borrowing from a creditor. The creditor offers technology-specific loan contracts to the firm, after which the firm makes its technology choice and subsequent decisions. Capital market imperfections impose financing frictions on the firm. Our analysis contributes to the capacity investment literature by extending the theory of stochastic capacity investment and flexible versus dedicated technology choice to understand the impact of capital market imperfections, and by analyzing the impact of demand uncertainty (variability and correlation) on the operational decisions and the performance of the firm under different capital market conditions. We demonstrate that the endogenous nature of credit terms in imperfect capital markets may modify or reverse conclusions concerning capacity investment and technology choice obtained under the perfect market assumption and we explain why. The theory developed in this paper suggests some rules of thumb for the strategic management of the capacity and technology choice in imperfect capital markets.

Key Words: Capacity, Flexibility, Financing, Newsvendor, Limited Liability, Market Imperfection.

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1 Introduction and Literature Review

Capacity investment is subject to internal or external financing frictions, especially in capital-intensive industries. If the internal capital of the firm is not sufficient to finance the desired investment level, then the firm may decide to raise external capital. External capital is more expensive because there exist capital market imperfections such as bankruptcy costs, taxes, financial distress cost or agency costs due to asymmetric information etc. (Froot et al. 1993) that create frictions in the borrowing process of the firm. However, as highlighted by Van Mieghem (2003, p. 275) “stochastic capacity models assume (often implicitly) [...] perfect capital markets, so that frictionless borrowing is possible [...].” In imperfect capital markets, the investment decision and the cost of external capital are interdependent. The objective of this paper is to increase our understanding of how capital market imperfections affect stochastic capacity investment and technology choice. A key feature of our paper is that we endogenize the cost of borrowing in a creditor-firm equilibrium.

To this end, we model a firm who produces and sells two products under demand uncertainty. The firm chooses between flexible and dedicated technologies that incur variable investment costs, and determines the capacity level and the production quantities with the chosen technology. Differing from the majority of the stochastic capacity investment literature, we assume that the firm is budget-constrained and can relax its budget constraint by borrowing from a creditor. The creditor offers technology-specific loan contracts to the firm, after which the firm makes its technology choice and subsequent decisions. We assume that the creditor incurs a fixed deadweight cost of financing if the firm defaults on the loan. In the basic model, we assume that the credit market is perfectly competitive. At the other end of the spectrum, we analyze a credit market with a monopolist creditor. In summary, the fixed bankruptcy cost and the monopolistic nature of the credit market constitute the capital market imperfections considered in this paper.

We derive the technology choice and external borrowing, capacity, and production level decisions of the firm and the creditor’s loan terms in equilibrium. We investigate how the demand uncertainty affects the capacity investment level and the performance of the firm in equilibrium for a given technology. We delineate the main drivers of the equilibrium technology choice and the impact of demand uncertainty on this choice. Our analysis focuses on determining the differences between perfect and imperfect capital markets; and understanding the impact of firm characteristics and different capital market conditions. We note that the objective of this paper is not to solve for the optimal capital structure of the firm (equity versus debt financing with different contractual terms); rather we focus on

technology-specific loan contracts characterized by their unit financing cost, and analyze the creditor-firm strategic interaction in that setting. Our results contribute to several streams of research, as detailed below.

The stochastic capacity investment literature analyzes the capacity-pooling value of flexible technology over dedicated technology and the impact of demand uncertainty on this value in a variety of models. We refer readers to Van Mieghem (2003) for an excellent review. As highlighted in this review paper, the operations management literature (often implicitly) assumes that capital markets are perfect, in which case operational and financial decisions decouple (Modigliani and Miller 1958). In practice, capital market imperfections exist (Harris and Raviv 1991) and impose deadweight costs of external financing, leading operational and financial decisions to interact with each other. This is because these financing costs are affected by the firm's operational decisions, and are endogenously determined in equilibrium. There is a growing body of work in operations and finance that analyzes these interactions. Our paper's overall contribution to this literature is to extend the theory of stochastic capacity investment and flexible versus dedicated technology choice to understand the impact of capital market imperfections. We focus on the impact of demand uncertainty on the capacity investment decision and the performance of the firm. We show that this impact takes different forms depending on the firm's loan type in equilibrium (secured with or without default possibility, unsecured) and the capital market conditions (perfectly competitive versus monopolistic credit market).

In the Operations Management literature, a recent stream of papers (Lederer and Singhal 1988, Buzacott and Zhang 2004, Xu and Birge 2004, Babich and Sobel 2004, Babich et al. 2006, Dada and Hu 2008 and Caldentey and Haugh 2009) analyze the joint financing and operating decisions of the firm and demonstrate the value of integrated decision making. All these papers focus on a single-product setting where technology choice is not relevant. An analytical characterization of the two-product firm is entirely new to this literature. The only paper to focus on flexible technology is Lederer and Singhal (1994), whose main focus is to study the joint financing (optimal mix of debt and equity) and capacity investment problem in a single-product, multi-period setting under the assumption of a perfectly competitive credit market. In a numerical example, they analyze the capacity-pooling benefit of flexible technology in a multi-product firm. They show that the value of flexible technology increases with a lower demand variability, and argue that this is because the default risk of the firm decreases, which allows the firm to secure a lower financing cost in equilibrium. Our analysis demonstrates that this result is only valid at high demand correlations. At low

demand correlations, the default risk of the firm is not affected by the change in demand variability because the diversification benefit of operating in two markets (that we call “financial pooling”) and the capacity-pooling benefit are sufficiently large. It follows that at low demand correlations, the value of flexible technology increases in demand variability. This is because i) the value of flexible technology at a given financing cost increases in demand variability (due to capacity pooling), and ii) the equilibrium level of financing cost is insensitive to changes in demand variability (due to financial- and capacity pooling).

Several finance papers also investigate the interaction of financing and operational decisions. Dotan and Ravid (1985) and Dammon and Senbet (1988) are examples of early research that demonstrates the effect of operational investments on the financing policy of the firm in a single-period setting. We refer the reader to Childs et al. (2005) for a recent review of papers in this stream. Among these, Mauer and Triantis (1994), Mello et al. (1995), and Mello and Parsons (2000) analyze the effect of various forms of operational flexibility (e.g. shutting down the production plant) on the joint operational and financing decisions of a firm in the contingent claims framework. The main focus of these papers is on financial issues; and therefore they have strong modeling assumptions concerning the firm’s operations. We demonstrate that new trade-offs arise and new insights are obtained with a more detailed formalization of the firm’s operations (the sequential nature of technology choice, capacity investment and production decisions) and the modeling of demand uncertainty. For example, MacKay (2003) empirically documents the negative correlation between production flexibility and the leverage of the firm. He attributes this to a higher equilibrium financing cost with production flexibility as the creditor believes that production flexibility will increase the riskiness of the firm’s cash flows. Our analysis demonstrates other facets of the interaction between production flexibility (flexible technology in our case) and the equilibrium credit terms based on a stronger formalization of the firm’s operations. We show that the equilibrium technology choice is determined through the interplay among the value of the limited liability option of the firm (only with an unsecured loan) and the financial-pooling benefit that exist with both technologies, and the capacity-pooling benefit that only exists with the flexible technology. Demand variability and correlation play a key role in the equilibrium technology choice through their impact on the relative effectiveness of these drivers.

The remainder of this paper is organized as follows: In §2, we describe the model and discuss the basis for our assumptions. §3, §4 and §5 focus on the perfectly competitive credit market case to derive the equilibrium strategy for a given technology, to investigate

the impact of the demand uncertainty on the firm’s capacity investment decision and expected equity value in equilibrium with each technology, and to investigate the equilibrium technology choice. In §6, we analyze the monopolistic credit market case and contrast it to the perfectly competitive credit market case. We conclude in §7 with a discussion of the limitations of our analysis and future research including potential avenues for empirical research.

2 Model Description and Assumptions

We consider a creditor-firm interaction where borrowing terms are determined before the firm makes any decisions. The firm is a budget-constrained monopolist whose shareholders have limited liability. The firm’s objective is to maximize the expected shareholder wealth by maximizing the expected value of equity. We model the firm’s decisions as a two-stage stochastic recourse problem: The firm makes its technology choice (dedicated D versus flexible F) and capacity investment decision (potentially after borrowing from the creditor) under demand uncertainty; and produces and sells two products after the resolution of this uncertainty. For consistency across scenarios, we focus on a stylized firm that lives for a single period and is liquidated at the end of the period. After operating profits are realized, the firm pays back its debt (if any), and default occurs if it is unable to do so. The sequence of events is presented in Figure 1.

Turning now to the creditor, we assume that the creditor offers a technology-specific unit financing cost a_T for $T \in \{D, F\}$ to the firm. At the time of contracting, the creditor has the same information as the firm. In determining the financing cost, the creditor takes into account not only the firm’s expected operating profits, but also the value P of any collateralized physical assets (e.g. real estate). For generality and tractability, we assume that the physical assets are illiquid - they can only be liquidated with a lead time. We assume $P \geq 0$. In the $P > 0$ case, default can occur due to the illiquidity of the physical assets even if the loan is secured. For the $P = 0$ case, there is no liquidity issue, and default occurs only due to insolvency.

As discussed in Froot et al. (1993), there exist deadweight costs of external financing that give rise to capital market imperfections. Paralleling this argument, we assume that the creditor incurs a fixed deadweight cost of financing S if the firm defaults on its loan; this cost is incurred as an out-of-pocket fee. S may represent *i*) the verification costs incurred by the creditor to observe the operating profits when the firm defaults (see, for example, Townsend (1979) and Froot et al. (1993)); *ii*) the opportunity cost of a foregone outside investment or the penalty cost due to an unfulfilled payment by the creditor to a third

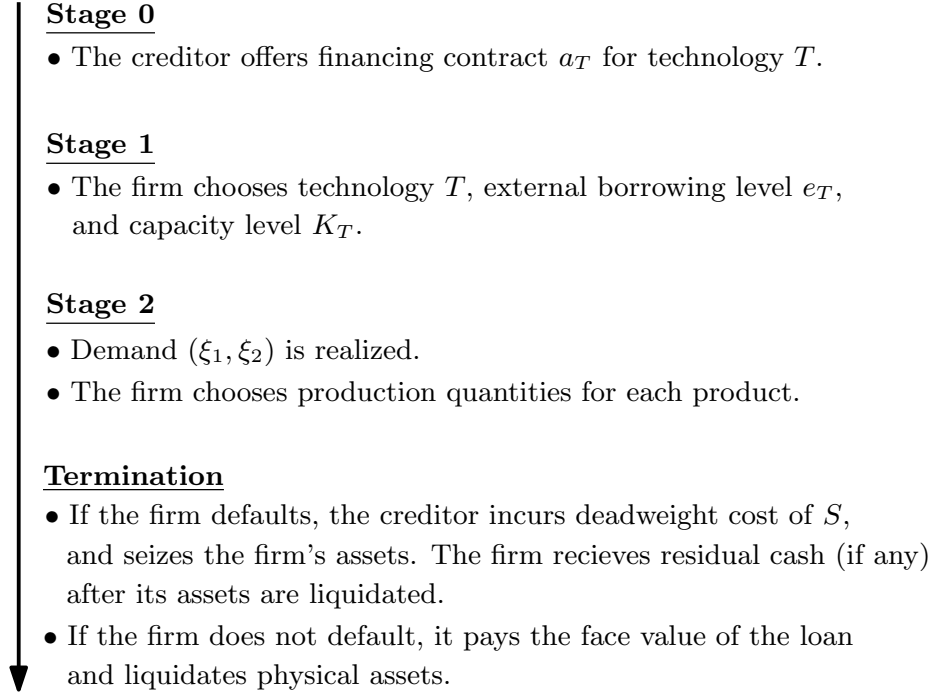


Figure 1: Timeline of events

party; *iii*) the transaction costs incurred by the creditor in the liquidation of the physical assets in case of default (Tirole 2006, p. 143); or *iv*) the direct cost of bankruptcy to the creditor¹, which includes the administrative and legal fees of the bankruptcy process (Altman 1980). Regardless of the interpretation of S , the existence of deadweight costs of financing introduces a capital market imperfection in our model.

In our basic model, the credit market is perfectly competitive and the creditor makes zero expected return. This is the common assumption used in the financial economics literature (e.g., Melnik and Plaut 1986). At the other end of the spectrum, we analyze a monopolist creditor who maximizes his expected return from lending and summarize our results in §6. In summary, the fixed deadweight cost of financing and the monopolistic nature of the credit market constitute the capital market imperfections considered in this paper.

¹With the fixed bankruptcy cost interpretation of S , when the bankruptcy is due to a liquidity issue (the firm can cover the face value of the loan using P , but defaults due to the illiquidity of the physical assets), our model implicitly assumes that the creditor does not wait for the firm to liquidate its assets and incurs the bankruptcy cost. We can prove that the resulting financing cost is higher than that offered if the creditor planned to wait and did not incur opportunity or penalty costs as a result. In other words, the creditor penalizes the firm in the financing rate looking forward to the possibility of bankruptcy due to a liquidity issue.

Returning to the timeline, before the firm makes any decisions, the creditor offers its borrowing terms $a_T \geq 0$, $T \in \{D, F\}$. The risk-free rate r_f is normalized to 0. In stage 1, the firm determines its technology choice $T \in \{D, F\}$, capacity investment level and borrowing level under the corresponding financing contract a_T with respect to the internal budget constraint B . The flexible technology (F) has a single resource that is capable of producing two products and the dedicated technology (D) consists of two resources that can each produce a single product. Thus, the flexible technology has a capacity-pooling benefit over the dedicated technology. Technology T incurs unit capacity investment cost c_T . Capacity investment can be salvaged at a rate of $0 \leq \gamma_T < 1$. Since flexible capacity is typically more marketable than dedicated capacity, we assume $\gamma_F \geq \gamma_D$.

In stage 2, demand uncertainty is resolved. The firm then chooses the production quantities (equivalently, prices) to satisfy demand optimally. Price-dependent demand for each product is represented by the iso-elastic inverse-demand function $p_i(q_i; \xi_i) = \xi_i q_i^{1/b}$ for $i = 1, 2$. Here, $b \in (-\infty, -1)$ is the constant price elasticity of demand, and p and q denote price and quantity, respectively. ξ_i represents the idiosyncratic risk component. We make specific assumptions about the distribution of (ξ_1, ξ_2) whenever necessary. In particular, we assume that $\boldsymbol{\xi}' = (\xi_1, \xi_2)$ has a symmetric bivariate normal distribution (with $\bar{\xi}_1 = \bar{\xi}_2 = \bar{\xi}$ and covariance matrix $\boldsymbol{\Sigma}$, where $\Sigma_{ii} = \sigma^2$ and $\Sigma_{ij} = \rho\sigma^2$ for $i \neq j$) because this is the natural setting to study the impact of demand correlation ρ . In this case, we use the convention that the support of the marginal distribution of ξ_i is characterized by $[\xi^l, \xi^u]$ where $\xi^l = \bar{\xi} - 3\sigma$ and $\xi^u = \bar{\xi} + 3\sigma$ as almost all of the probability mass is located in this range. We assume that the marginal production costs of each product are 0. This is an assumption that is widely used in the literature for tractability (see Goyal and Netessine 2007 and references therein).

After operating profits are realized, the firm salvages its capacity investment. If the firm is able to repay its debt from its final cash position (that consists of operating revenues and the salvage value of capacity), it does so and, since the firm lives for a single period, terminates by liquidating its physical assets. Otherwise, default occurs. The cash on hand and the ownership of the collateralized physical assets (if any) are transferred to the creditor. The creditor may or may not be able to retrieve the face value of the loan from the seized assets of the firm depending on whether the firm is solvent or not when the magnitude of P is taken into account. In the former case, the firm collects the remaining cash.

We use the following mathematical representation throughout the text: A realization of the random variable ξ is denoted by $\tilde{\xi}$ and its expectation is denoted by $\bar{\xi}$. Bold face

letters represent vectors of the required size. Vectors are column vectors and $'$ denotes the transpose operator. \mathbf{x}^a denotes the componentwise exponent a of the vector \mathbf{x} . \mathbf{xy} denotes the componentwise product of vectors \mathbf{x} and \mathbf{y} with identical dimensions. Pr denotes probability, \mathbb{E} denotes the expectation operator, A^c denotes the complement of set A and $(x)^+ \doteq \max(x, 0)$. Monotonic relations (increasing, decreasing) are used in the weak sense unless otherwise stated.

3 The Equilibrium Strategy

3.1 Dedicated Technology

In this section, we characterize the equilibrium decisions of the firm and the creditor when the firm uses the dedicated technology. All the proofs in the paper are relegated to a Technical Appendix that can be found on the authors' web sites. The supporting technical statements, denoted by A.x, are also provided in the Technical Appendix. Since we assume a symmetric distribution for $\boldsymbol{\xi}$, the firm optimally invests in identical capacity levels for both resources. Therefore, we can use a single resource level K_D to characterize the firm's optimal capacity investment portfolio.

3.1.1 Analysis of the Firm's Problem for a Given Financing Cost

In this section, we describe the optimal solution for the firm's capacity investment, external borrowing and production decisions using backward induction starting from stage 2.

Stage 2, Production Decision: In stage 1, the firm with budget B borrowed e_D , invested in capacity level K_D for each resource and placed $B + e_D - 2c_D K_D$ into the cash account (at the risk-free rate). In this stage, the firm observes the demand realizations $\tilde{\boldsymbol{\xi}} = (\tilde{\xi}_1, \tilde{\xi}_2)$ and determines the production quantities $\mathbf{q}' = (q_1, q_2)$ within the existing capacity limit K_D for each product to maximize the stage-two equity value $\Pi_D(\mathbf{q}; K_D, e_D, B, \tilde{\boldsymbol{\xi}})$. To derive Π_D , note that two outcomes are possible: If the firm's final cash position (consisting of stage-two operating profits, cash account holdings and the salvage value of capacity) is sufficient to cover the face value of the loan, i.e. if $\mathbf{q}' \mathbf{p}(\mathbf{q}; \tilde{\boldsymbol{\xi}}) + (B + e_D - 2c_D K_D) + 2\gamma_D c_D K_D \geq e_D(1 + a_D)$, then the firm does not default. Otherwise, it defaults and its assets (including the ownership of physical assets P) are transferred to the creditor. The deadweight cost of financing S is borne by the creditor as an out-of-pocket expenditure. The firm receives the remaining cash (if any) after the face value of the loan is deducted from its seized assets. With the limited liability assumption, we can therefore write

$$\Pi_D(\mathbf{q}; K_D, e_D, B, \tilde{\boldsymbol{\xi}}) = \left[\mathbf{q}' \mathbf{p}(\mathbf{q}; \tilde{\boldsymbol{\xi}}) + (B + e_D - 2c_D K_D) + 2\gamma_D c_D K_D - e_D(1 + a_D) \right]^+. \quad (1)$$

Maximizing the stage-two equity value is equivalent to maximizing the operating profit. We find $q_1^* = q_2^* = K_D$, in other words, the firm optimally utilizes all of its available capacity for each product. Then the optimal equity value is given by $\Pi_D^*(K_D, e_D, B, \tilde{\xi}) = \left[(\tilde{\xi}_1 + \tilde{\xi}_2) K_D^{(1+\frac{1}{b})} + (B + e_D - 2c_D K_D) + 2\gamma_D c_D K_D + P - e_D(1 + a_D) \right]^+$.

Stage 1, Capacity Choice and External Financing: In this stage, the firm has an internal budget $B \geq 0$ and determines the optimal capacity investment level K_D^* for each resource and the optimal external borrowing level e_D^* so as to maximize its expected equity value, $\pi_D(K_D, e_D; B) = \mathbb{E} \left[\Pi_D^*(K_D, e_D, B, \tilde{\xi}) \right]$. It is easy to show that at optimality, $e_D = (2c_D K_D - B)^+$ is satisfied, that is, the firm exactly borrows what it needs to cover its capacity investment. Thus, the optimal expected equity value of the firm, $\pi_D^*(B)$, can be found as follows:

$$\max_{K_D \geq 0} \mathbb{E} \left[(\tilde{\xi}_1 + \tilde{\xi}_2) K_D^{(1+\frac{1}{b})} + (B - 2c_D K_D)^+ + 2\gamma_D c_D K_D + P - (2c_D K_D - B)^+(1 + a_D) \right]^+. \quad (2)$$

For a given capacity investment level K_D , if the firm has borrowed, it does not default when demand is such that $\tilde{\xi}_1 + \tilde{\xi}_2 \geq d_D(K_D) \doteq (1 + a_D - \gamma_D) 2c_D K_D^{-\frac{1}{b}} - B(1 + a_D) K_D^{-(1+\frac{1}{b})}$, while it defaults, but is able to pay back the loan after its collateralized assets are liquidated when $d_D(K_D) > \tilde{\xi}_1 + \tilde{\xi}_2 \geq l_D(K_D) \doteq d_D(K_D) - P \times K_D^{-(1+\frac{1}{b})}$. We call $d_D(K_D)$ and $l_D(K_D)$ the *default threshold* and the *limited liability threshold*, respectively, for investment level K_D with the dedicated technology. If the firm does not have any physical assets to collateralize ($P = 0$), then the default threshold equals the limited liability threshold.

It is easy to establish that $l_D(K_D)$ is strictly increasing in K_D . We define K_D^l as the (unique) solution to $l_D(K_D^l) = 2\xi^l$, where ξ^l is the lower bound on demand for each product. For $K_D \leq K_D^l$, the bracketed expression in (2) is non-negative for any demand realization $\tilde{\xi}$, i.e. the loan is secured. For $K_D > K_D^l$, for some $\tilde{\xi}$, the bracketed expression is negative, i.e. the loan is unsecured. The objective function in (2) is strictly concave for $K_D \in [0, K_D^l]$, but is not necessarily globally concave in K_D . Note that as the bracketed expression in (2) becomes more negative, not being liable for negative cash flows becomes more valuable. In this case, we say that the value of the limited liability option of the firm increases.

Proposition 1 For the firm with an internal budget $B \geq B_D^h \doteq 2c_D \hat{K}_D \left[1 - \frac{\xi^l}{\xi^u(1+\frac{1}{b})} \right] \left[1 - \frac{\gamma_D}{1+a_D} \right] - \frac{P}{1+a_D}$ where $\hat{K}_D \doteq \left(\frac{\xi^u(1+\frac{1}{b})}{(1+a_D-\gamma_D)c_D} \right)^{-b}$, the unique K_D^* and the expected equity value π_D^* are

given by

$$K_D^* = \begin{cases} K_D^0 \doteq \left(\frac{\bar{\xi}(1+\frac{1}{b})}{(1-\gamma_D)c_D} \right)^{-b} & \text{if } B \geq 2c_D K_D^0 \\ \frac{B}{2c_D} & \text{if } 2c_D K_D^1 \leq B < 2c_D K_D^0 \\ K_D^1 \doteq \left(\frac{\bar{\xi}(1+\frac{1}{b})}{(1+a_D-\gamma_D)c_D} \right)^{-b} & \text{if } B < 2c_D K_D^1, \end{cases}$$

$$\pi_D^* = \begin{cases} \frac{2c_D K_D^0(1-\gamma_D)}{-(b+1)} + B + P & \text{if } B \geq 2c_D K_D^0 \\ 2\bar{\xi} \frac{B}{2c_D} + \gamma_D B + P & \text{if } 2c_D K_D^1 \leq B < 2c_D K_D^0 \\ \frac{2c_D K_D^1(1+a_D-\gamma_D)}{-(b+1)} + B(1+a_D) + P & \text{if } B < 2c_D K_D^1 \end{cases}$$

The condition $B \geq B_D^h$ ensures that the objective function is globally concave and the optimal investment level can be found in closed form². Here, K_D^0 is the budget-unconstrained capacity level and K_D^1 is the capacity investment level with borrowing if the loan needed to make this investment is secured. If the budget realization is high enough to cover the corresponding cost $2c_D K_D^0$, then $K_D^* = K_D^0$ with no borrowing. For $2c_D K_D^1 \leq B < 2c_D K_D^0$, the budget is insufficient to cover $2c_D K_D^0$, and the marginal revenue of capacity is lower than its marginal cost with borrowing. Therefore, the firm invests in the capacity level $\frac{B}{2c_D}$ that fully utilizes its internal budget. For $B < 2c_D K_D^1$, the firm borrows to invest in capacity. In this case, since the internal budget level is sufficiently high ($B \geq B_D^h$), the loan is secured.

Proposition 2 For the firm with $B < B_D^l \doteq 2c_D K_D^1 \left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})} \right] \left[1 - \frac{\gamma_D}{1+a_D} \right] - \frac{P}{1+a_D}$, $K_D^* \in (K_D^1, \hat{K}_D)$ is a solution to $MP_D(K_D^*) = 0$, where

$$MP_D(K_D) = \int \int_{\Upsilon_D(\boldsymbol{\xi}; K_D)} \left[\left(1 + \frac{1}{b} \right) (\xi_1 + \xi_2) K_D^{\left(\frac{1}{b}\right)} - 2(1 + a_D - \gamma_D)c_D \right] f(\xi_1, \xi_2) d\xi_1 d\xi_2,$$

with $\Upsilon_D(\boldsymbol{\xi}; K_D) \doteq \{\boldsymbol{\xi} : \xi_1 + \xi_2 \geq l_D(K_D)\}$ and $f(\xi_1, \xi_2)$ is the joint pdf of $\boldsymbol{\xi}$. The optimal expected equity value is given by

$$\pi_D^* = \int \int_{\Upsilon_D(\boldsymbol{\xi}; K_D^*)} \left[(\tilde{\xi}_1 + \tilde{\xi}_2) K_D^{*(1+\frac{1}{b})} - 2c_D K_D^*(1 + a_D - \gamma_D) + B(1 + a_D) + P \right] f(\xi_1, \xi_2) d\xi_1 d\xi_2.$$

Since the internal budget level is low ($B < B_D^l$), the firm borrows and uses an unsecured loan. The optimal solution is not necessarily unique, as neither global concavity nor unimodality can be guaranteed. It can be shown that in this budget range the firm would use an unsecured loan to invest in K_D^1 . Therefore, the firm optimally takes more investment risk with borrowing, and increases K_D^* beyond K_D^1 .

²For $P \geq \bar{P}_D \doteq 2c_D^{b+1} (\xi^u(1+\frac{1}{b}))^{-b} \left(1 - \frac{\xi^l}{\xi^u(1+\frac{1}{b})} \right) (1-\gamma_D)^{b+1}$, we have $B_D^h < 0$ for any $a_D \geq 0$. In other words, Proposition 1 characterizes the optimal solution for any $B \geq 0$ and $a_D \geq 0$ if P is large enough.

For the firm with $B \in [B_D^l, B_D^h)$, we cannot explicitly characterize the optimal capacity investment level for a general distribution of ξ . However, we prove that if $b \geq -2$ and ξ follows a bivariate normal distribution, then π_D is unimodal.

Proposition 3 *If π_D is unimodal in K_D , then the unique K_D^* is given by*

$$K_D^* = \begin{cases} K_D^0 & \text{if } B \geq 2c_D K_D^0 \\ \frac{B}{2c_D} & \text{if } 2c_D K_D^1 \leq B < 2c_D K_D^0 \\ K_D^1 & \text{if } 2c_D K_D^1 \left[1 - \frac{\xi^l}{\xi(1+\frac{1}{b})} \right] \left[1 - \frac{\gamma_D}{1+a_D} \right] - \frac{P}{1+a_D} \leq B < 2c_D K_D^1 \\ \bar{K}_D & \text{if } 0 \leq B < 2c_D K_D^1 \left[1 - \frac{\xi^l}{\xi(1+\frac{1}{b})} \right] \left[1 - \frac{\gamma_D}{1+a_D} \right] - \frac{P}{1+a_D} \end{cases} \quad (3)$$

where $\bar{K}_D \in (K_D^1, \hat{K}_D)$ is the unique solution to $MP_D(\bar{K}_D) = 0$. The optimal expected equity value π_D^* decreases in a_D .

The intuition of the first two cases in (3) is similar to Proposition 1. If the budget is large enough such that the firm can invest in K_D^1 using a secured loan (the third row), then the optimal capacity level with borrowing is K_D^1 . If the budget is sufficiently low such that the firm would use an unsecured loan to invest in K_D^1 , then the firm optimally takes more investment risk and the optimal capacity investment level with borrowing is $\bar{K}_D > K_D^1$.

With the bivariate normal distribution assumption, $MP_D(K_D)$ is characterized by

$$\left[1 - \Phi \left(\frac{l_D(K_D) - \bar{\mu}}{\bar{\sigma}} \right) \right] \left[\left(1 + \frac{1}{b} \right) \bar{\mu} K_D^{(\frac{1}{b})} - 2(1 + a_D - \gamma_D) c_D \right] + \left(1 + \frac{1}{b} \right) \bar{\sigma} K_D^{(\frac{1}{b})} \phi \left(\frac{l_D(K_D) - \bar{\mu}}{\bar{\sigma}} \right),$$

where $\bar{\mu} = 2\bar{\xi}$, $\bar{\sigma} = \sigma\sqrt{2(1+\rho)}$, and $\Phi(\cdot)$ and $\phi(\cdot)$ are the cdf and pdf of the standard normal random variable, respectively.

The optimal expected equity value of the firm, π_D^* , is as given in Proposition 1 when the firm does not borrow or it uses a secured loan. If the firm uses an unsecured loan, the optimal expected equity value of the firm (as adapted from Proposition 2 to the bivariate normal distribution) can be written as

$$\bar{\sigma} \bar{K}_D^{(1+\frac{1}{b})} \phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right) + \left[1 - \Phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right) \right] [\bar{\mu} K_D^{(1+\frac{1}{b})} - 2c_D(1 + a_D - \gamma_D) \bar{K}_D + B(1 + a_D) + P].$$

Since π_D^* decreases in a_D , the firm benefits from a lower financing cost. Throughout the remaining dedicated technology analysis, we assume that π_D is unimodal in K_D and use the characterization given in Proposition 3.

3.1.2 Characterization of the Creditor's Expected Return

Let $\Lambda_D(a_D)$ denote the creditor's expected return for a given unit financing cost a_D with the dedicated technology. When the firm borrows for a given $a_D \geq 0$, $\Lambda_D(a_D)$ is given by

$$\Lambda_D(a_D) = (2c_D K_D^*(a_D) - B) a_D - S \times F_D(d_D(K_D^*(a_D))) - L_D(K_D^*(a_D)) \quad (4)$$

where $K_D^*(a_D)$ denotes the optimal capacity level of the firm for a given a_D , $F_D(x) \doteq Pr(\xi_1 + \xi_2 \leq x)$ and $L_D(K_D^*(a_D))$ is given as

$$\int \int_{\Upsilon_D^c(\xi; K_D^*(a_D))} \left[(2c_D K_D^*(a_D) - B)(1 + a_D) - (\tilde{\xi}_1 + \tilde{\xi}_2) K_D^*(a_D)^{(1+\frac{1}{b})} - 2\gamma_D c_D K_D^*(a_D) - P \right] f(\xi_1, \xi_2) d\xi_1 d\xi_2.$$

In (4), the first term is the creditor's net gain from lending if the loan is secured by the collateralized assets of the firm, the second term denotes the expected deadweight cost of financing (payable when the firm defaults), and the third term is the expected loss due to the unsecured part of the loan.

Proposition 4 *If the firm's problem is unimodal, then the creditor's expected return $\Lambda_D(a_D)$ is characterized by*

$$\begin{aligned} i) &= (2c_D K_D^1 - B)a_D && \text{for } 0 \leq a_D < a_D^{max} && \text{if } \begin{aligned} &2c_D K_D^0 (1 - \gamma_D) \frac{(\bar{\xi}(1+\frac{1}{b}) - \xi^l)}{\bar{\xi}(1+\frac{1}{b})} \leq B, \\ &< 2c_D K_D^0 \end{aligned} \\ ii) &= \begin{cases} (2c_D K_D^1 - B)a_D - F_D(d_D(K_D^1))S & \text{for } 0 \leq a_D < a_D^d \\ (2c_D K_D^1 - B)a_D & \text{for } a_D^d \leq a_D < a_D^{max} \end{cases} && \text{if } \begin{aligned} &2c_D K_D^0 (1 - \gamma_D) \frac{(\bar{\xi}(1+\frac{1}{b}) - \xi^l)}{\bar{\xi}(1+\frac{1}{b})} - P \leq \\ &B < 2c_D K_D^0 (1 - \gamma_D) \frac{(\bar{\xi}(1+\frac{1}{b}) - \xi^l)}{\bar{\xi}(1+\frac{1}{b})} \end{aligned} \\ iii) &= \begin{cases} (2c_D \bar{K}_D - B)a_D - F_D(d_D(\bar{K}_D))S - L_D(\bar{K}_D) & \text{for } 0 \leq a_D < a_D^l \\ (2c_D K_D^1 - B)a_D - F_D(d_D(K_D^1))S & \text{for } a_D^l \leq a_D < a_D^d \\ (2c_D K_D^1 - B)a_D & \text{for } a_D^d \leq a_D < a_D^{max} \end{cases} && \text{if } B < 2c_D K_D^0 (1 - \gamma_D) \frac{(\bar{\xi}(1+\frac{1}{b}) - \xi^l)}{\bar{\xi}(1+\frac{1}{b})} - P, \end{aligned}$$

where $a_D^{max} \doteq \left[\left(\frac{2c_D K_D^0}{B} \right)^{-\frac{1}{b}} - 1 \right] (1 - \gamma_D)$, a_D^l , the unsecured loan threshold, is the unique solution to $B = 2c_D K_D^0 (1 - \gamma_D)^{-b} \frac{(\bar{\xi}(1+\frac{1}{b}) - \xi^l)}{\bar{\xi}(1+\frac{1}{b})} \frac{(1+a_D^l - \gamma_D)^{(b+1)}}{(1+a_D^l)} - \frac{P}{1+a_D^l}$, and a_D^d , the secured loan with default possibility threshold, is the unique solution to $B = 2c_D K_D^0 (1 - \gamma_D)^{-b} \frac{(\bar{\xi}(1+\frac{1}{b}) - \xi^l)}{\bar{\xi}(1+\frac{1}{b})} \frac{(1+a_D^d - \gamma_D)^{(b+1)}}{(1+a_D^d)}$.

If $B \geq 2c_D K_D^0$, the firm does not borrow for any $a_D \geq 0$, and the creditor does not have any returns (this is omitted from the statement of the proposition). Otherwise, the firm borrows if the financing cost is not very high, i.e. for $a_D \in [0, a_D^{max}]^3$. If the internal budget level is sufficiently high (case (i)), the firm borrows to invest in $K_D^1(a_D)$ but never defaults for any $a_D \in [0, a_D^{max}]$. This case can only occur if there is a positive lower bound on demand or a positive salvage value of capacity. If the internal budget level is moderate (case (ii)), for a small a_D , the firm borrows to invest in $K_D^1(a_D)$ and may default on the loan, but the creditor can always retrieve the face value of the loan through the collateralized assets. For large a_D , the firm borrows less to invest in $K_D^1(a_D)$ and does not default. In summary, in case (ii), the firm may default but the borrowing is always secured. If the internal budget level is sufficiently low (case (iii)), the firm uses an unsecured loan to invest in $\bar{K}_D(a_D)$ for small a_D . For moderate a_D , the firm borrows less to invest in $K_D^1(a_D)$, may default, but the loan is secured. For large a_D , the firm borrows even less and does not default.

³ $a_D < a_D^{max}$ is equivalent to $B < 2c_D K_D^1$ in Proposition 3.

If $b \geq -2$ and ξ follows a bivariate normal distribution, as discussed in §3.1.1, the firm's problem is unimodal and the characterization in Proposition 4 holds. In this case, we have $F_D(d_D(K_D^*(a))) = \Phi\left(\frac{d_D(K_D^*(a)) - \bar{\mu}}{\bar{\sigma}}\right)$ where $\bar{\mu} = 2\bar{\xi}$, $\bar{\sigma} = \sigma\sqrt{2(1+\rho)}$ and $\Phi(\cdot)$ is cdf of the standard normal random variable. The expected loss for a given \bar{K}_D , i.e. $L_D(\bar{K}_D)$, can also be characterized in closed-form in a similar fashion as $\pi_D^*(\bar{K}_D)$. If P is sufficiently large, i.e. $P \geq \bar{P}_D$, only cases (i) and (ii) of Proposition 4 are relevant, and the bivariate normal distribution and $b \geq -2$ assumptions are not required.

3.1.3 Equilibrium Characterization

We now turn to the characterization of the equilibrium. We use the \dot{x} notation to denote equilibrium quantities: \dot{a}_D is the equilibrium unit financing cost and $\dot{K}_D \doteq K_D^*(\dot{a}_D)$, $\dot{\pi}_D = \pi_D^*(\dot{a}_D)$ are the equilibrium capacity investment level and expected equity value of the firm respectively. When there are multiple a_D 's that satisfy the objective of the creditor, we set \dot{a}_D to the smallest such value. \dot{a}_D is Pareto-optimal for the firm because its optimal expected equity value increases as a_D decreases.

For the perfectly competitive credit market, we can show that if the firm's problem is unimodal, i.e. if $b \geq -2$ and ξ follows a bivariate normal distribution or if $P \geq \bar{P}_D$, $\dot{a}_D = 0$ for case (i) of Proposition 4 and $\dot{a}_D \in (0, a_D^d)$ otherwise. In other words, if the firm does not default for any feasible a_D , then the creditor's expected return is always non-negative. In this case, $\Lambda_D(a_D) = 0$ is achieved at $\dot{a}_D = 0$. If the firm may default for some a_D , then $\dot{a}_D < a_D^d$, i.e. the equilibrium financing cost is such that the firm has a positive default probability. In this case, either of the two equilibria, $0 < \dot{a}_D < a_D^l$ or $a_D^l \leq \dot{a}_D < a_D^d$, may arise.

In summary, depending on the credit market, firm and product market characteristics, one of the following three types of equilibria will be observed: *an equilibrium where the firm uses a secured loan without default possibility*, i.e. $a_D^d \leq \dot{a}_D < a_D^{max}$, *an equilibrium where the firm uses a secured loan with default possibility*, i.e. $a_D^l \leq \dot{a}_D < a_D^d$, and *an equilibrium where the firm uses an unsecured loan*, i.e. $\dot{a}_D < a_D^l$. When the firm uses a secured loan, it invests in $K_D^1(\dot{a}_D)$, whereas when the firm uses an unsecured loan, it invests in $\bar{K}_D(\dot{a}_D)$. For $P \geq \bar{P}_D$, the firm always uses a secured loan in equilibrium and only the first two equilibria are observed.

3.2 Flexible Technology

The analysis is similar to the dedicated technology analysis with minor modifications. For brevity, we only provide a synopsis of the analysis. We first analyze the firm's decision problem for a given financing cost a_F . We can characterize a B_F^h such that for $B \geq B_F^h$,

the optimal capacity investment level is given by an analogue of Proposition 1 with two modifications: The optimal capacity investment level that uses the internal budget is $\frac{B}{c_F}$; and in K_F^0 and K_F^1 , the term $\bar{\xi}$ is replaced by $\mathbb{E} \left[\left(\xi_1^{-b} + \xi_2^{-b} \right)^{-\frac{1}{b}} \right]$. This new term captures the capacity-pooling benefit of the flexible technology. In this case, when the firm borrows, the loan is secured and the firm invests in K_F^1 . We can also characterize a B_F^l such that for $B < B_F^l$, the optimal capacity investment level is given by an analogue of Proposition 2: \bar{K}_F is a solution to $MP_F(\bar{K}_F) = 0$, where

$$MP_F(K_F) = \int \int_{\Upsilon_F(\boldsymbol{\xi}; K_F)} \left[\left(1 + \frac{1}{b} \right) \left(\xi_1^{-b} + \xi_2^{-b} \right)^{-\frac{1}{b}} K_F^{\left(\frac{1}{b}\right)} - (1 + a_F - \gamma_F) c_F \right] f(\xi_1, \xi_2) d\xi_1 d\xi_2,$$

with $\Upsilon_F(\boldsymbol{\xi}; K_F) \doteq \left\{ \boldsymbol{\xi} : \left(\xi_1^{-b} + \xi_2^{-b} \right)^{-\frac{1}{b}} \geq l_F(K_F) \right\}$, and $l_F(K_F) \doteq K_F^{-\frac{1}{b}} (1 + a_F - \gamma_F) c_F - K_F^{(-1-\frac{1}{b})} [B(1 + a_F) + P]$ is the limited liability threshold with the flexible technology. In this case, since the firm uses an unsecured loan, the firm optimally takes more investment risk with borrowing, and we have $\bar{K}_F > K_F^1$. For $B \in [B_F^l, B_F^h)$, there is no analytical characterization of K_F^* for a general $\boldsymbol{\xi}$ distribution. Unlike the dedicated technology analysis, the unimodality of the firm's expected equity value π_F in K_F cannot be proven with the bivariate normal distribution of $\boldsymbol{\xi}$. Therefore, for an arbitrary value of P , we cannot obtain an analogue of Proposition 3 with the flexible technology, and we resort to numerical experiments. In these experiments, we observe that the firm's equity value π_F is unimodal in K_F , and \bar{K}_F is uniquely characterized when the firm uses an unsecured loan. For a sufficiently large P , i.e. $P \geq \bar{P}_F \doteq 2c_F^{b+1} [\xi^u (1 + \frac{1}{b})]^{-b} \left[1 - \frac{\xi^l}{\xi^u (1 + \frac{1}{b})} \right] [1 - \gamma_F]^{b+1}$, we can show that the firm uses a secured loan with borrowing, and the optimal capacity investment level is as given in an analogue of Proposition 1.

For the creditor's problem, if we assume a sufficiently large P , i.e. $P > \bar{P}_F$, the creditor's expected return as a function of a_F can be characterized in a similar fashion to Proposition 4 where only cases (i) and (ii) are relevant. For an arbitrary value of P , we cannot guarantee the existence and uniqueness of the a_F thresholds a_F^l and a_F^d paralleling those in Proposition 4. Thus, the structure of the creditor's expected return given in case (iii) does not need to hold. In this case, we resort to numerical experiments to characterize the creditor's expected return. Since we observe the unimodality of π_F in our numerical experiments, the characterization given in case (iii) continues to hold numerically.

The characterization of the unique Pareto-optimal equilibrium \dot{a}_F for flexible technology is similar to the dedicated technology case. For $P \geq \bar{P}_F$, there exist two different equilibria: An equilibrium where the firm uses a secured loan without default possibility ($\dot{a}_F = 0$) and

an equilibrium where the firm uses a secured loan with default possibility ($\dot{a}_F \in (0, a_F^d)$). For an arbitrary value of P , we observe the three different equilibria paralleling the dedicated technology case.

4 The Impact of Demand Uncertainty

The goal of this paper is to analyze the impact of endogenous credit terms under capital market imperfections in a capacity investment setting. To this end, we first identify the perfect capital market equilibrium with a given technology $T \in \{D, F\}$.

If the capital markets are perfect, there is no deadweight cost of financing ($S = 0$). In this case, the firm's capacity investment decision is independent of the financing decision:

Remark 1 *In the perfect capital market equilibrium, for any firm with $B \geq 0$, we have $\dot{K}_T = K_T^0 = \left(\frac{M_T(1+\frac{1}{b})}{(1-\gamma_T)c_T} \right)^{-b}$ and the expected equity value of the firm is given by $\dot{\pi}_T = B + P + \frac{\eta_T K_T^0 (1-\gamma_T)}{-(b+1)}$ where $M_D = \bar{\xi}$, $M_F = \mathbb{E} \left[\left(\xi_1^{-b} + \xi_2^{-b} \right)^{-\frac{1}{b}} \right]$, $\eta_D = 2c_D$ and $\eta_F = c_F$.*

The equilibrium investment level with either technology is the budget-unconstrained investment level K_T^0 for the firm with any internal budget level as in traditional stochastic capacity models: The firm simply chooses the optimal investment level without regard to the budget limit, and implements it by borrowing if necessary. This replicates the well-known result about the decoupling of operational and financial decisions in perfect markets (Modigliani and Miller 1958); but we do it to have the benchmark specific to our model. We now show that the impact of the demand variability and demand correlation in perfect capital markets is modified once capital market imperfections and endogenous credit terms are taken into account, and we explain why.

The numerical experiments used in this section use the following data set: $b = -2$, $c_T \in \{3, 3.25, 3.5, 3.75, 4\}$, $\bar{\xi} \in \{10, 12.5, 15, 17.5, 20\}$, $P \in \{0, 125, 250\}$, $S \in \{5, 25, 50, 75, 150, 200\}$ and $B \in \{0.5, 2.5, 5, 7.5, 10\}$. For brevity, we also normalize the salvage rate with each technology (γ_T) to zero. We assume that ξ follows a bivariate normal distribution. To analyze the impact of demand variability, we use the mean-preserving spread of the normal distribution. For a given mean $\bar{\xi}$, a higher σ leads to a higher demand variability. We use $\sigma \in [14\%, 30\%]$ of $\bar{\xi}$ with 2%-unit increments⁴. To analyze the impact of demand correlation, we focus on $\rho \in \{-0.9995, -0.75, -0.5, -0.25, 0, 0.25, 0.5, 0.75, 0.9995\}$. In summary, we investigate 2250 numerical instances at each of the 81 (σ, ρ) combinations for both of the

⁴This set of σ values implies that the coefficient of variation is not large, hence the non-negativity of ξ embodied in our normality assumption is unproblematic.

technologies. For convenience, we summarize the results of this section in Table 1, where the boxed comparative statics results are proven analytically and the rest are existence results observed in numerical experiments.

	Perfect Market		Perfectly Competitive Credit Market			
	Form of \dot{K}_T	Effect	Form of \dot{K}_T	Effect	Form of \dot{K}_T	Effect
σ	K_D^0 K_F^0	$\dot{K}_{D-}, \dot{\pi}_{D-}$ $\dot{K}_F \downarrow, \dot{\pi}_F \downarrow$	$K_D^1(\dot{a}_D)$ secured w/o default possibility	$\dot{K}_{D-}, \dot{\pi}_{D-}, \dot{a}_{D-}$	$K_F^1(\dot{a}_F)$ secured w/o default possibility	$\dot{K}_F \downarrow, \dot{\pi}_F \downarrow, \dot{a}_F -$
			$K_D^1(\dot{a}_D)$ secured w default possibility	$\dot{K}_D \downarrow, \dot{\pi}_D \downarrow, \dot{a}_D \uparrow$	$K_F^1(\dot{a}_F)$ secured w default possibility	$\dot{K}_F \uparrow \downarrow, \dot{\pi}_F \uparrow \downarrow, \dot{a}_F \uparrow$
			$\bar{K}_D(\dot{a}_D)$ unsecured	$\dot{K}_D \uparrow \downarrow, \dot{\pi}_D \downarrow, \dot{a}_D \uparrow$	$\bar{K}_F(\dot{a}_F)$ unsecured	$\dot{K}_F \uparrow \downarrow, \dot{\pi}_F \uparrow \downarrow, \dot{a}_F \uparrow$
ρ	K_D^0 K_F^0	$\dot{K}_{D-}, \dot{\pi}_{D-}$ $\dot{K}_F \uparrow, \dot{\pi}_F \uparrow$	$K_D^1(\dot{a}_D)$ secured w/o default possibility	$\dot{K}_{D-}, \dot{\pi}_{D-}, \dot{a}_{D-}$	$K_F^1(\dot{a}_F)$ secured w/o default possibility	$\dot{K}_F \downarrow, \dot{\pi}_F \downarrow, \dot{a}_F -$
			$K_D^1(\dot{a}_D)$ secured w default possibility	$\dot{K}_D \downarrow, \dot{\pi}_D \downarrow, \dot{a}_D \uparrow$	$K_F^1(\dot{a}_F)$ secured w default possibility	$\dot{K}_F \downarrow, \dot{\pi}_F \downarrow, \dot{a}_F \uparrow$
			$\bar{K}_D(\dot{a}_D)$ unsecured	$\dot{K}_D \uparrow \downarrow, \dot{\pi}_D \downarrow, \dot{a}_D \uparrow$	$\bar{K}_F(\dot{a}_F)$ unsecured	$\dot{K}_F \downarrow, \dot{\pi}_F \downarrow, \dot{a}_F \uparrow$

Table 1: Differences on the impact of a (local) increase in demand variability σ and demand correlation ρ between perfect and imperfect markets with bivariate normal demand uncertainty for a perfectly competitive credit market. The boxed results are proven analytically. $-$ denotes no change, \uparrow denotes an increase and \downarrow denotes a decrease.

4.1 Dedicated Technology

In perfect capital markets, as follows from Remark 1, the firm's equilibrium capacity level for each resource and the expected equity value with the dedicated technology do not depend on the demand variability σ or the demand correlation ρ . In imperfect capital markets, this result continues to hold at equilibria where the firm uses a secured loan without default possibility (Rows 1 and 4 of the ‘‘Dedicated Technology’’ column of Table 1). In the other two cases, \dot{K}_D and $\dot{\pi}_D$ depend on σ and ρ .

Proposition 5 (Rows 2 and 5 of Table 1) *At equilibria where the firm uses a secured loan with default possibility, \dot{a}_D increases while $\dot{\pi}_D$ and \dot{K}_D decrease in σ or ρ (locally).*

From the firm's perspective, for an arbitrary a_D , the demand variability or the demand correlation does not alter the optimal capacity investment level or the equity value (as the firm uses a secured loan). Therefore, the impact of σ or ρ is only determined by its effect on \dot{a}_D . For a bivariate normal $\boldsymbol{\xi}$, the aggregate demand $\xi_1 + \xi_2$ has a normal distribution with mean $\bar{\mu} = 2\bar{\xi}$ and standard deviation $\bar{\sigma} = \sigma\sqrt{2(1+\rho)}$. Therefore, an increase in σ or ρ increases the aggregate demand variability $\bar{\sigma}$, thus, increases the downside risk, and in turn, the default risk of the firm. This increases the expected deadweight cost of financing and decreases the expected return of the creditor for an arbitrary a_D . Therefore, the rate \dot{a}_D increases. Since \dot{a}_D increases, \dot{K}_D and $\dot{\pi}_D$ decrease.

At equilibria where the firm uses an unsecured loan (Rows 3 and 6 of Table 1), the impact of demand variability and demand correlation are determined through the interplay between two effects: *i*) the value of the limited liability option of the firm and *ii*) the equilibrium financing cost. For a given a_D , an increase in σ or ρ increases the aggregate demand variability $\bar{\sigma}$, and in turn, $K_D^*(a) = \bar{K}_D(a_D)$ and $\pi_D^*(a_D)$ (Lemma A.3). This is because as the likelihood of low demand states increases, the value of the limited liability option of the firm increases. For a given $\bar{\sigma}$, a lower unit financing cost increases $\bar{K}_D(a_D)$ (Lemma A.3) and $\pi_D^*(a_D)$ (as follows from Proposition 3). Therefore, if an increase in σ or ρ decreases \dot{a}_D , then we conclude that \dot{K}_D and $\dot{\pi}_D$ increase since both effects work in the same direction; otherwise, the result depends on which effect dominates. We now analyze the impact of σ and ρ on \dot{a}_D .

In a perfectly competitive credit market, from the creditor's perspective, increasing σ or ρ , thus increasing aggregate demand variability $\bar{\sigma}$ has three distinct effects corresponding to the three components of $\Lambda_D(a_D)$. As we show in Lemma A.4, the default risk⁵ and the expected loss due to the unsecured part of the loan increase (as for a fixed \bar{K}_D , the downside risk of the firm's operating cash flows increases, and the firm borrows more as \bar{K}_D increases); and the creditor's net gain increases (the firm borrows more as \bar{K}_D decreases). The first two effects work to increase \dot{a}_D , whereas the third effect works to decrease it. In all of our numerical experiments, we observe that the first two effects dominate and \dot{a}_D increases with an increase in σ or ρ . To determine whether the increase in \dot{a}_D or the increase in the value of limited liability option of the firm dominates, we resort to numerical experiments. For $\dot{\pi}_D$, we observe that the former dominates and $\dot{\pi}_D$ decreases with an increase in σ or ρ . For \dot{K}_D , we observe that the former effect dominates in the majority of the numerical instances and \dot{K}_D decreases with an increase in σ or ρ . For some instances, which correspond to moderate

⁵Lemma A.4 proves this with an additional condition which is satisfied in our numerical experiments.

aggregate demand variability $\bar{\sigma}$ levels, the latter effect dominates and \dot{K}_D increases.

The above discussion applies for local sensitivity analysis. With a sufficiently large increase in σ or ρ , the unique equilibrium may switch from a secured loan without default possibility to a secured loan with default possibility, and then to an unsecured loan (if any). In the first transition, \dot{a}_D becomes strictly positive, thus \dot{K}_D and $\dot{\pi}_D$ decrease with an increase in σ or ρ . In the second transition, we observe in our numerical experiments that \dot{K}_D and $\dot{\pi}_D$ decrease.

In summary, the impact of $\bar{\sigma}$ is determined through the interplay between the value of the limited liability option of the firm (only with an unsecured loan) and the equilibrium financing cost. We find that:

1. The value of the limited liability option of the firm increases with an increase in $\bar{\sigma}$.
2. The impact of $\bar{\sigma}$ on \dot{a}_D is through its impact on the default risk with a secured loan.

This impact is multi-dimensional with an unsecured loan, as the capacity investment level is affected by the aggregate demand variability due to limited liability.

3. An increase in $\bar{\sigma}$ increases \dot{a}_D . Therefore, the impact of an increase in $\bar{\sigma}$ is determined by the trade-off between increasing \dot{a}_D and increasing the value of the limited liability option (if any):

- a. For $\dot{\pi}_D$, the equilibrium financing cost is the main determinant, and an increase in $\bar{\sigma}$ decreases $\dot{\pi}_D$.

- b. For \dot{K}_D , the equilibrium financing cost is the main determinant with a secured loan or with an unsecured loan when $\bar{\sigma}$ is sufficiently high: An increase in $\bar{\sigma}$ decreases \dot{K}_D . The value of the limited liability option of the firm is the main determinant when $\bar{\sigma}$ is moderate: An increase in $\bar{\sigma}$ increases \dot{K}_D . This is because \dot{a}_D is not very sensitive to changes in $\bar{\sigma}$ in this range. At lower $\bar{\sigma}$ values, there is no default possibility and \dot{K}_D does not change in $\bar{\sigma}$.

In turn, the aggregate demand variability $\bar{\sigma}$ depends directly on σ and ρ . A higher demand variability σ increases the aggregate demand variability. The impact of demand correlation ρ follows from a *financial-pooling* argument: Operating in two markets creates a diversification benefit for the firm, i.e. the variability of aggregate demand, $\bar{\sigma}^2$, is lower than the sum of the variability of the individual demands, $2\sigma^2$. As ρ increases, the financial-pooling benefit decreases as the firm generates similar revenues from both markets.

We conclude this section by discussing an interesting interaction between the demand correlation and the demand variability that we observe in our numerical experiments: At low σ levels, the firm uses a secured loan without default possibility and the impact of σ or ρ on \dot{K}_D and $\dot{\pi}_D$ is zero; and this is observed at all ρ levels. The interaction between σ

and ρ is more subtle at higher σ levels. For a given ρ , there exists a threshold value of σ , $\tilde{\sigma}_D(\rho)$, below which $\dot{a}_D \approx 0$ and is insensitive to ρ and σ . Consequently, the insensitivity of \dot{K}_D or $\dot{\pi}_D$ to σ , which is observed at secured loan equilibria without default possibility, is now observed at secured loan equilibria with default possibility and even unsecured loan equilibria. This is due to the financial pooling phenomenon that makes $\bar{\sigma}$ small relative to σ for sufficiently low values of ρ such that the default probability becomes negligible. As ρ increases, the financial-pooling benefit decreases and $\tilde{\sigma}_D(\rho)$ decreases.

4.2 Flexible Technology

In perfect capital markets, as follows from Remark 1, the firm's equilibrium capacity level and the expected equity value with the flexible technology depend on the demand variability (σ) and the demand correlation (ρ) through the term $M_F = E \left[\left(\xi_1^{-b} + \xi_2^{-b} \right)^{-\frac{1}{b}} \right]$. This term captures the capacity-pooling feature of flexible technology. Unfortunately, it is not possible to derive analytically the effect of ρ and σ on M_F for bivariate normal $\boldsymbol{\xi}$. To derive the analytical results for flexible technology in Table 1, we assume that M_F is decreasing in ρ and increasing in σ . This assumption is in line with the traditional argument on flexible technology investment: Its value increases in demand variability and decreases in demand correlation. In our numerical experiments, consistent with this assumption, we observe that M_F increases with a higher σ or a lower ρ . Therefore, \dot{K}_F and $\dot{\pi}_F$ increase in perfect capital markets as M_F increases.

In imperfect capital markets, the impact of the demand variability (σ) and the demand correlation (ρ) with the flexible technology is determined through the interplay among the value of capacity pooling, the value of the limited liability option of the firm (only with an unsecured loan) and the equilibrium financing cost. An increase in ρ or a decrease in σ decreases the value of capacity pooling for a given a_F . An increase in ρ or σ increases the value of the limited liability option of the firm for a given a_F as the likelihood of low demand states increases.

For the impact of σ and ρ on \dot{a}_F , σ and ρ directly affect the default risk and the expected loss due to the unsecured part of the loan, and indirectly affect the net gain from secured lending, the default risk and the expected loss due to the unsecured part of the loan by changing the capacity investment level. In our numerical experiments, we observe that the net effect is such that \dot{a}_F increases with an increase in σ or ρ . This is illustrated in Panel A of Figure 2 for the secured loan case.

For the impact of ρ on \dot{K}_F and $\dot{\pi}_F$, the value of capacity pooling is the main determinant (both with a secured and an unsecured loan): An increase in ρ decreases \dot{K}_F and $\dot{\pi}_F$. This

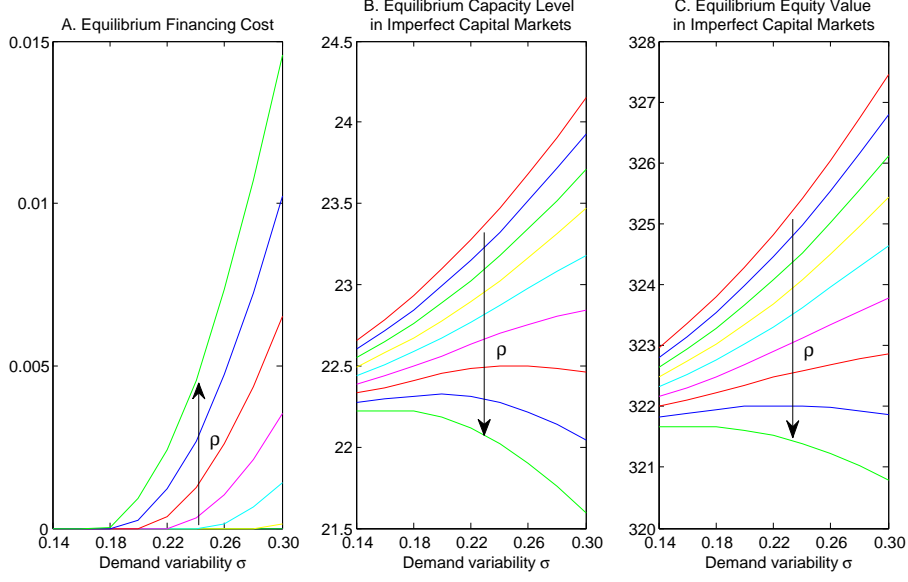


Figure 2: Effect of demand variability σ and demand correlation ρ on flexible technology with $S = 25$, $B = 5$, $c_F = 3$, $\bar{\mu} = 20$ and $P = 250$: In a perfectly competitive credit market, a higher ρ or a higher σ increases \dot{a}_F (Panel A). With an increase in ρ , \dot{K}_F (Panel B) and $\dot{\pi}_F$ (Panel C) decrease. A higher σ increases (decreases) \dot{K}_F (Panel B) and $\dot{\pi}_F$ (Panel C) at low (high) σ levels.

can be observed from Panels B and C of Figure 2, respectively.

For the impact of σ on the same, we observe that for a given ρ , there exists a threshold value of σ , $\tilde{\sigma}_F(\rho)$, below which $\dot{a}_F \approx 0$ and is insensitive to σ . This is due to the financial-pooling and the capacity-pooling benefits that make the default probability negligible. As ρ increases, these benefits decrease and $\tilde{\sigma}_F(\rho)$ decreases. This can be observed from Panel A of Figure 2. When (σ, ρ) are such that $\sigma \leq \tilde{\sigma}_F(\rho)$, the sole determinant is capacity pooling such that \dot{K}_F and $\dot{\pi}_F$ increase with an increase in σ . When $\sigma > \tilde{\sigma}_F(\rho)$, we have $\dot{a}_F > 0$, and the impact of σ on \dot{K}_F and $\dot{\pi}_F$ is more subtle and depends on ρ . If ρ is not high, the value of capacity pooling and the value of the limited liability option (only with an unsecured loan) are the main determinants such that \dot{K}_F and $\dot{\pi}_F$ increase in σ . This is because \dot{a}_F is not sensitive to σ due to the financial- and the capacity-pooling benefits. At high ρ levels, the equilibrium financing cost starts dominating such that \dot{K}_F and $\dot{\pi}_F$ decrease in σ . This is because \dot{a}_F is very sensitive to σ as the financial- and capacity-pooling benefits are low, and the increase in \dot{a}_F outweighs the increase in the already low values of capacity pooling and the value of the limited liability option of the firm. These effects are depicted in Panels B and C of Figure 2 for the secured loan case.

In summary, the differences between the flexible and the dedicated technologies in Table

1 are due to the capacity-pooling feature of the flexible technology. Besides its value for a given financing cost a_F , which is similar to the perfect capital market benchmark case, capacity pooling has a strategic value in imperfect capital markets as it is one of the main determinants of \dot{a}_F . This strategic value has interesting implications for the equilibrium technology choice in imperfect capital markets as we illustrate in the next section.

5 Technology Choice

In §4, we investigated the impact of demand uncertainty (σ and ρ) for a given technology. In this section, we investigate the equilibrium technology choice in imperfect markets and how this choice is affected by σ or ρ compared to the technology choice in perfect capital markets. Since the firm always invests in a positive level of capacity with each technology, investing in either technology dominates not making any technology investment. We first characterize the equilibrium technology choice in perfect capital markets.

Remark 2 *If the capital markets are perfect, there exists a unique variable cost threshold $\bar{c}_D^P(c_F)$ such that when $c_D \leq \bar{c}_D^P(c_F)$, it is optimal to invest in dedicated technology. This threshold is given by*

$$\bar{c}_D^P(c_F) \doteq c_F \left(\frac{1 - \gamma_F}{1 - \gamma_D} \right) \left[\frac{2^{-\frac{1}{b}} \bar{\xi}}{\mathbb{E} \left[\left(\xi_1^{-b} + \xi_2^{-b} \right)^{-\frac{1}{b}} \right]} \right]^{\frac{b}{b+1}} \leq c_F,$$

where the last inequality holds at equality only if the salvage values are symmetric ($\gamma_F = \gamma_D$) and the demands are deterministic ($\sigma = 0$) or perfectly positively correlated ($\rho = 1$).

The threshold $\bar{c}_D^P(c_F)$ is a variant of the flexibility premium of Chod et al. (2009). Since flexible capacity has a higher salvage value and has a capacity-pooling benefit, we have $\bar{c}_D^P(c_F) \leq c_F$. As we discussed in §4, the term $E \left[\left(\xi_1^{-b} + \xi_2^{-b} \right)^{-\frac{1}{b}} \right]$ captures the capacity-pooling benefit of the flexible technology. This threshold decreases with an increase in demand variability (σ) and a decrease in demand correlation (ρ) due to the increasing capacity-pooling benefit.

We now investigate the equilibrium technology choice in imperfect capital markets. For the numerical experiments in this section, we use the same data set as in §4. At each of the 81 (σ, ρ) combinations that we consider, we calculate an imperfect capital market cost threshold $\bar{c}_D^I(c_F)$ that makes the firm indifferent between the dedicated and the flexible technologies for each of the 2250 numerical instances⁶. We also numerically calculate the perfect market

⁶We cannot prove the uniqueness of $\bar{c}_D^I(c_F)$ because \dot{a}_D is not monotone in c_D , and a higher capacity investment cost c_D may result in a lower financing cost \dot{a}_D . In that case, the impact of an increase in c_D on

$\sigma \backslash \rho$	-0.9995	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	0.9995
14%	N	N	N	N	N	N	(1.87, 100)	(12.4, 100)	(28, 100)
16%	N	N	N	N	N	(6.53, 100)	(26.4, 100)	(46.67, 100)	(62.53, 100)
18%	N	N	N	N	(7.2, 100)	(32.93, 100)	(56.93, 100)	(73.47, 100)	(84, 100)
20%	N	N	N	(2.67, 100)	(31.2, 100)	(60.53, 100)	(78.98, 100)	(88.31, 100)	(94.27, 100)
22%	N	N	N	(18.4, 100)	(56.67, 100)	(79.6, 100)	(89.87, 100)	(96.27, 100)	(98.13, 100)
24%	N	N	(0.67, 100)	(40.13, 100)	(76.04, 100)	(89.73, 100)	(96.53, 100)	(98.67, 100)	(99.47, 100)
26%	N	N	(10.93, 100)	(62.8, 100)	(86.67, 100)	(96.4, 100)	(99.07, 100)	(99.6, 100)	(100, 100)
28%	N	N	(28.4, 100)	(78.18, 100)	(93.96, 92.05)	(98.53, 91.88)	(99.51, 92.85)	(99.91, 93.24)	(99.82, 92.97)
30%	N	N	(48.4, 100)	(86.8, 100)	(97.24, 88.44)	(99.16, 83.91)	(99.78, 83.34)	(99.78, 83.47)	(99.69, 79.98)

Table 2: Summary of the difference between $\bar{c}_D^I(c_F)$ and $\bar{c}_D^P(c_F)$ with respect to demand variability (σ) and demand correlation (ρ) in a perfectly competitive credit market. At each (σ, ρ) combination, first entry denotes the percentage of numerical instances in which this difference is non-zero, i.e. one of the technologies is favored in imperfect capital markets compared to the perfect market benchmark. The second entry denotes the percentage of the numerical instances out of the non-zero instances in which the difference is negative, i.e. flexible technology is favored. When the difference is zero for all of the numerical instances, we write “N.” For each of the (σ, ρ) combinations, we calculate $\bar{c}_D^I(c_F)$ and $\bar{c}_D^P(c_F)$ for 2250 numerical instances with $b = -2$, $c_D \in \{3, 3.25, 3.5, 3.75, 4\}$, $\bar{\xi} \in \{10, 12.5, 15, 17.5, 20\}$, $P \in \{0, 125, 250\}$, $S \in \{5, 25, 50, 75, 150, 200\}$, $B \in \{0.5, 2.5, 5, 7.5, 10\}$ and $\gamma_F = \gamma_D = 0$.

cost threshold $\bar{c}_D^P(c_F)$ at these instances and analyze the difference between $\bar{c}_D^I(c_F)$ and $\bar{c}_D^P(c_F)$. At any numerical instance with a given c_F , if $\bar{c}_D^I(c_F) < (>) \bar{c}_D^P(c_F)$, then flexible (dedicated) technology is chosen for a larger (smaller) set of c_D levels in the imperfect capital market compared to the perfect market case. In that case, we say that flexible (Dedicated) technology is “favored” in imperfect capital markets. If $\bar{c}_D^I(c_F) = \bar{c}_D^P(c_F)$, we say neither technology is favored.

Our main results are summarized in Table 2. At a given (σ, ρ) combination, if we do not observe any numerical instance with a non-zero difference between the two cost thresholds, we use “N”. Otherwise we report two numbers: The first one is the percentage of numerical instances (out of the 2250) where we observe a non-zero difference between $\bar{c}_D^I(c_F)$ and $\bar{c}_D^P(c_F)$. The second one is the percentage of numerical instances out of the non-zero observations in which flexible technology is favored. For example, at $\sigma = 30\%$ and $\rho = 0.75$, in 99.78% of the numerical instances, we observe a non-zero difference between the two cost thresholds, and in 83.47% of these non-zero instances, flexible technology is favored (and in the remaining 16.53% of instances, dedicated technology is favored).

$\bar{\pi}_D$ is ambiguous. However, in all of our numerical experiments, we observe a unique $\bar{c}_D^I(c_F)$.

In Table 2, we observe three “bands,” one where neither technology is favored, one where flexible technology is always favored, and one where either technology may be favored⁷. To delineate the intuition behind this observation, we will focus on a technology cost pair that is insightful to discuss, $(\bar{c}_D^P(c_F), c_F)$, where the firm is indifferent between the two technologies in perfect capital markets. The equilibrium technology choice in imperfect capital markets at this technology cost pair tells us which technology is favored in imperfect capital markets. We next analyze the equilibrium technology choice at this cost pair, and the impact of the demand variability and the demand correlation on this choice.

At low demand correlation levels or moderate demand correlation levels accompanied with sufficiently low demand variability levels, neither technology is favored. As discussed in §4, for a given ρ , there exists a threshold with each technology, $\tilde{\sigma}_T(\rho)$, below which $\dot{a}_T \approx 0$. This is due to the financial-pooling benefit, i.e. the aggregate demand variability $\bar{\sigma}$ is small relative to σ for sufficiently low values of ρ , and the default probability becomes negligible. When the financial-pooling benefit is high enough to avoid default with both technologies, neither technology is favored. This is observed at low ρ levels $(-0.9995, -0.75)$, and moderate ρ levels accompanied with sufficiently low σ levels ($\sigma \leq 0.22\bar{\mu}$ at $\rho = -0.5$, $\sigma \leq 0.18\bar{\mu}$ at $\rho = -0.25$, $\sigma \leq 0.16\bar{\mu}$ at $\rho = 0$ and $\sigma = 0.14\bar{\mu}$, $\rho = 0.25$ which are all equivalent to $\bar{\sigma} \leq 0.16\bar{\mu}$). As demand variability increases, the financial-pooling benefit decreases, and the equilibrium technology choice can be different from the perfect capital market benchmark case. This difference depends on the demand correlation and the type of loan the firm uses as we discuss next.

At moderate (σ, ρ) combinations, if a technology is favored, it is the flexible technology. At sufficiently high (σ, ρ) combinations, dedicated technology may be favored. In our numerical experiments, we observe that $\tilde{\sigma}_F(\rho) > \tilde{\sigma}_D(\rho)$, i.e. the default probability remains negligible, and $\dot{a}_T \approx 0$, for larger σ values for a given ρ with the flexible technology. This is because the flexible technology enjoys both financial- and capacity-pooling benefits, while the dedicated technology only enjoys the former. At (σ, ρ) combinations such that $\tilde{\sigma}_D(\rho) < \sigma \leq \tilde{\sigma}_F(\rho)$, we have $\dot{a}_D > \dot{a}_F = 0$. Therefore, the flexible technology is strictly preferred with the technology cost pair $(\bar{c}_D^P(c_F), c_F)$, i.e. the flexible technology is favored in imperfect capital markets. At (σ, ρ) combinations such that $\sigma > \tilde{\sigma}_F(\rho)$, \dot{a}_F and \dot{a}_D are both positive. In this case, the equilibrium technology choice depends on the type of loan the firm uses.

⁷Our numerical data set assumes $\gamma_F = \gamma_D = 0$. With $\gamma_F > \gamma_D$, we continue to observe three bands with the same properties as those depicted in Table 2.

When the firm uses a secured loan, we can show that at the technology cost pair $(\bar{c}_D^P(c_F), c_F)$, flexible (dedicated) technology is chosen if and only if $\dot{a}_F < \dot{a}_D$ ($\dot{a}_F > \dot{a}_D$). We can also show that the ordering of \dot{a}_D and \dot{a}_F is determined by the ordering of the default risk with identical financing costs, which in turn is determined by the trade-off between the capacity-pooling benefit of the flexible technology and the higher total capacity investment made under the dedicated technology. In our numerical experiments, we observe that the former effect dominates and the flexible technology is favored. We note here that $\dot{a}_F \leq \dot{a}_D$ is observed in all of our numerical experiments. This implies that with a secured loan, the (weak) dominance of flexible technology in imperfect capital markets holds at all (σ, ρ) combinations.

When the firm uses an unsecured loan in equilibrium, the value of the limited liability option of the firm with each technology matters. In this case, the technology choice at the technology cost pair $(\bar{c}_D^P(c_F), c_F)$ is not given by the ordering of \dot{a}_F and \dot{a}_D , but is determined by the relative magnitudes of \dot{a}_F and \dot{a}_D , the value of limited liability option of the firm with each technology, and the capacity-pooling benefit of the flexible technology. In our numerical experiments, at high (σ, ρ) combinations, we observe numerical instances in which dedicated technology is favored in imperfect capital markets. This is because at these (σ, ρ) levels, the value of the limited liability option of the firm is very high (as discussed in §4), and this is more beneficial for the dedicated technology (as the total capacity investment cost is higher with the dedicated technology). The higher value of the limited liability option may outweigh the lower equilibrium financing cost and the capacity-pooling benefit of the flexible technology. Indeed, instances where the dedicated technology dominates are observed at $\sigma \geq 0.28\bar{\mu}$ with $\rho \geq 0$.

In summary, the equilibrium technology choice in imperfect capital markets, and the impact of demand variability and demand correlation on this choice are determined through the interplay among the value of the limited liability option of the firm (only with an unsecured loan), the financial-pooling benefit that exists with both technologies, and the capacity-pooling benefit, that only exists with flexible technology.

1. At low demand correlation levels or moderate demand correlation levels accompanied with sufficiently low demand variability levels, the financial-pooling benefit is significantly high with both technologies and the default probability is negligible in equilibrium. Therefore, the equilibrium technology choice is identical to the perfect market benchmark case.

2. For higher σ and ρ levels, the equilibrium technology choice deviates from the perfect market benchmark:

- a. With a secured loan, this deviation is due to the impact of capacity pooling on the equilibrium financing cost: The capacity-pooling feature of the flexible technology induces the firm to secure a lower financing cost in equilibrium and flexible technology is favored.
- b. With an unsecured loan, this deviation is determined by the interplay between the capacity-pooling feature of the flexible technology and the value of the limited liability option of the firm with both technologies. At high demand variability levels accompanied with moderate-to-high demand correlation levels, dedicated technology may be favored due to the dominance of the limited liability option of the firm at these levels. Otherwise, flexible technology is favored due to its capacity-pooling benefit.

6 Extension: Monopolistic Credit Market

In this section, we analyze our problem in a monopolistic credit market, and discuss our results by making a comparison with the perfectly competitive credit market case. Since the firm's problem with a given financing cost and the characterization of the creditor's problem is not affected, we start with the equilibrium characterization for each technology.

In the monopolistic credit market, the unit financing cost is chosen so as to maximize the expected profit of the creditor, thus the equilibrium financing cost is larger than the one in the perfectly competitive credit market. This cost is characterized by equaling the *marginal* expected profit $\frac{\partial}{\partial a_T} \Lambda(a_T)$, and not the expected profit $\Lambda(a_T)$, to zero⁸. Therefore, the equilibrium financing cost is determined by the interplay among the marginal net gain from secured lending, the marginal default risk and the marginal expected loss due to the unsecured part of the loan. Similar to the perfectly competitive credit market case, one of the three types of equilibria is observed: an equilibrium where the firm uses *i*) a secured loan without default possibility, *ii*) a secured loan with default possibility, or *iii*) an unsecured loan.

With the dedicated technology, a local increase in $\bar{\sigma}$ has the same impact on \dot{K}_D and $\dot{\pi}_D$ with the perfectly competitive credit market case as summarized in Table 1. However, the intuition behind these results is different as the equilibrium financing cost is determined by the marginal expected profit of the creditor. For example, at equilibria where the firm uses a secured loan with default probability, the increase in \dot{a}_D is due to a decrease in the marginal default risk (and not due to an increase in the default risk). Similar to the

⁸If there are multiple global maximizers of the creditor's problem, then the lowest of these is chosen as the unique Pareto-optimal equilibrium. However, we never encounter such a case in our numerical instances.

$\sigma \backslash \rho$	-0.9995	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	0.9995
14%	N	N	N	N	N	N	(4.13, 100)	(13.6, 100)	(24.13, 100)
16%	N	N	N	N	N	(8.67, 100)	(22.27, 100)	(37.07, 100)	(50.53, 100)
18%	N	N	N	N	(9.56, 100)	(27.07, 100)	(44.58, 100)	(61.51, 99.42)	(73.69, 90.53)
20%	N	N	N	(4.8, 100)	(25.69, 100)	(47.56, 100)	(66.76, 96.67)	(79.2, 88.10)	(86.76, 79.76)
22%	N	N	N	(17.87, 100)	(43.91, 100)	(67.47, 93.61)	(81.29, 85.40)	(89.33, 79.60)	(94.09, 67.50)
24%	N	N	(3.02, 100)	(32.67, 100)	(63.07, 97.82)	(80.58, 86.98)	(89.6, 79.51)	(94.09, 79.22)	(95.56, 50.51)
26%	N	N	(11.91, 100)	(49.2, 99.91)	(76.36, 90.40)	(88.71, 82.01)	(94.18, 71.35)	(95.56, 71.26)	(95.91, 42.96)
28%	N	N	(24.04, 100)	(65.47, 96.40)	(85.64, 88.22)	(93.38, 83.20)	(94.93, 77.06)	(96.49, 64.90)	(96.31, 37.01)
30%	N	N	(37.11, 99.52)	(76.93, 94.17)	(91.11, 87.32)	(96.18, 77.96)	(96.67, 68.83)	(97.73, 62.35)	(96.04, 25.50)

Table 3: Summary of the difference between $\bar{c}_D^I(c_F)$ and $\bar{c}_D^P(c_F)$ with respect to demand variability (σ) and demand correlation (ρ) in a monopolistic credit market. For each of the (σ, ρ) combinations, we calculate $\bar{c}_D^I(c_F)$ and $\bar{c}_D^P(c_F)$ for the same 2250 numerical instances that we use in Table 2 ($b = -2$, $c_D \in \{3, 3.25, 3.5, 3.75, 4\}$, $\bar{\xi} \in \{10, 12.5, 15, 17.5, 20\}$, $P \in \{0, 125, 250\}$, $S \in \{5, 25, 50, 75, 150, 200\}$, $B \in \{0.5, 2.5, 5, 7.5, 10\}$ and $\gamma_F = \gamma_D = 0$).

perfectly competitive credit market case, we observe in our numerical experiments that with a sufficiently large increase in $\bar{\sigma}$, the unique equilibrium may switch from a secured loan without default possibility to a secured loan with default possibility, and then to an unsecured loan (if any). Unlike the perfectly competitive credit market case, in the second transition, \dot{a}_D may decrease and \dot{K}_D and $\dot{\pi}_D$ may increase. This happens when the creditor's expected return is bimodal: The increase in $\bar{\sigma}$ may induce the creditor to switch from one local maximizer (in the secured lending region) to the other local maximizer (in the unsecured lending region). This leads to a discontinuous decrease in \dot{a}_D such that \dot{K}_D and $\dot{\pi}_D$ increase.

With the flexible technology, we continue to observe \dot{a}_F increasing in σ or ρ in the majority of our numerical experiments, in which case, all the comparative static results of Table 1 continue to hold. In some numerical instances, however, \dot{a}_F decreases with an increase in σ or ρ . In this case, there is an interesting modification to the results of Table 1: With an increase in ρ , when \dot{a}_F decreases, the lowering of the financing cost may outweigh the lowering of the capacity-pooling benefit and $\dot{\pi}_F$ may increase. This is observed at equilibria where the firm uses an unsecured loan or a secured loan with default possibility.

For the equilibrium technology choice in a monopolistic credit market, Table 3, which is an analogue of Table 2, summarizes our results.

The band of (σ, ρ) levels in Table 3 where neither technology is favored is identical to the perfectly competitive credit market case. In contrast, the band of (σ, ρ) levels in Table 3 where dedicated technology may be favored is much larger than Table 2. This is because, unlike the perfectly competitive credit market case, the equilibrium financing cost may be *lower* with the dedicated technology. We now explain the intuition behind this result, once

again focusing on the technology cost pair $(\bar{c}_D^P(c_F), c_F)$.

When the firm uses a secured loan with both technologies, as discussed in §5, at the technology cost pair $(\bar{c}_D^P(c_F), c_F)$, flexible (dedicated) technology is chosen if and only if $\dot{a}_F < \dot{a}_D$ ($\dot{a}_F > \dot{a}_D$). In a monopolistic credit market, the ordering of \dot{a}_F and \dot{a}_D is determined by the ordering of the marginal default risk with each technology, i.e. the rate of reduction in the default probability with an increase in a_T . The impact of the capacity-pooling benefit of the flexible technology on the marginal default risk is indeterminate. In particular, the marginal default risk may be lower and in turn, the equilibrium financing cost may be higher with the flexible technology. Therefore, dedicated technology may be favored in equilibrium. In a perfectly competitive credit market, the impact of the capacity-pooling benefit of the flexible technology on the default risk is determinate: The default risk is lower and in turn, the equilibrium financing cost is lower with the flexible technology. Therefore, dedicated technology is never favored with a secured loan in a perfectly competitive credit market.

When the firm uses an unsecured loan, the value of the limited liability option of the firm with each technology matters for a given financing cost and also impacts the equilibrium financing cost. In our numerical experiments, we continue to observe $\dot{a}_D < \dot{a}_F$ in several instances. The lowering of the equilibrium financing cost with the dedicated technology adds to the higher value of the limited liability option of the firm with the dedicated technology, and the dedicated technology is favored at a larger set of numerical instances as compared to the perfectly competitive credit market case.

In summary, in a monopolistic credit market, the drivers of the equilibrium technology choice and the impact of demand uncertainty are identical to the perfectly competitive credit market case. For a given financing cost, these drivers work in the same direction, however, their impacts on the equilibrium financing cost may be different. The most significant consequence of this difference is that the dedicated technology is more prevalent in the monopolistic credit market equilibrium.

7 Conclusion

This paper contributes to the stochastic capacity investment literature by relaxing the (often implicit) perfect capital market assumption and analyzing the impact of endogenous credit terms under capital market imperfections. A joint operational and financial perspective is adopted to develop theory and insights into capacity management and technology choice in imperfect capital markets. In a two-product setting, we analyze the impact of the demand uncertainty (variability and correlation) on the capacity investment decision and

the expected equity value of the firm in equilibrium with dedicated and flexible technology investments, as well as the choice between flexible and dedicated technology in equilibrium. Except for a numerical analysis in Lederer and Singhal (1994), there is no formal treatment of the two-product firm in the literature, therefore, the two-product analysis is a distinct contribution of our research.

In a perfect capital market, the dedicated technology investment level is not affected by demand uncertainty, whereas flexible technology is affected due to its capacity-pooling feature. In an imperfect capital market, we show that demand uncertainty matters with the dedicated technology. In particular, the impact of the demand variability and correlation in equilibrium is determined through the interplay between the value of the limited liability option of the firm (only with an unsecured loan) and the equilibrium financing cost. With the flexible technology, there is a third facet in this interplay, the capacity-pooling benefit. Our results are summarized in Table 1 and in §6, and demonstrate that the impact of demand uncertainty in imperfect capital markets can be different from the perfect capital market benchmark, and that these comparative statics results depend on the firm's loan type in equilibrium (unsecured versus secured, with or without default possibility) and the different capital market conditions studied.

In a perfect capital market, the equilibrium technology choice, as well as the impact of demand variability and demand correlation on this choice, are determined by the capacity-pooling feature of the flexible technology. In imperfect capital markets, what matters is the interplay among the value of the limited liability option of the firm (with an unsecured loan) and the financial-pooling benefit (the diversification benefit from operating in two markets) that exist with both technologies, and the capacity-pooling benefit that only exists with the flexible technology. We show that introducing capital market imperfections (weakly) favors one of the technologies with respect to the perfect capital market benchmark, and this preference depends on the demand variability and demand correlation. Our results are summarized in Tables 2 and 3 and demonstrate that at high demand variability and correlation levels dedicated technology may be favored, otherwise flexible technology is weakly favored. Although the general pattern of the equilibrium technology choice is the same for the capital market conditions studied, the strength of the preference depends on these conditions.

This paper brings constructs and assumptions motivated by the finance literature into a classical operations management problem and develops new insights. In turn, using a stronger formalization of operational decisions than in the finance literature (the sequen-

tial nature of technology choice, capacity investment and production decisions) and the modeling of demand uncertainty, we provide novel insights on issues discussed in this literature. For example, Melnik and Plaut (1986) derive several relations among the parameters of loan contracts based on the assumption that the borrowing level is independent of the unit financing cost and that the default probability increases in the unit financing cost. Our analysis demonstrates that these assumptions may not be valid with a more formal representation of operational decisions: The firm optimally adjusts its capacity investment level; thus the borrowing level may decrease and the default probability increase in the unit financing cost. As argued in MacKay (2003), firms with higher operational flexibility are assumed to have a lower default risk due to the option value of operational flexibility. Our analysis shows that this argument acquires new dimensions with a stronger formalization of the firm's operations: Anticipating the option value of operational flexibility (the capacity-pooling benefit of the flexible technology), the firm optimally adjusts the other operational decisions (capacity investment level), and the default risk in equilibrium changes.

Our summaries at the end of each section suggest some rules of thumb for the strategic management of the capacity investment portfolio and technology choice and provide the basis for potential empirical research in this domain. While a formal development of empirical hypotheses is beyond the scope of this paper, the following predictions of our model would be interesting to explore empirically:

1. The higher the demand variability or demand correlation, the lower will be the performance of firms using dedicated technology when the credit market is highly competitive.
2. The higher the demand variability, the lower (higher) will be the capacity investment and performance of firms using flexible technology when the demand correlation is high (low) when the credit market is highly competitive.
3. The higher the demand correlation, the lower will be the capacity investment and performance of firms using flexible technology when the credit market is highly competitive.
4. The prevalence of flexible technology choice is higher for firms using a secured loan than for firms using an unsecured loan regardless of the competitiveness of the credit market.
5. The higher the demand variability, the higher the prevalence of firms using dedicated technology regardless of the competitiveness of the credit market.

There are a number of limitations to the present study that lead to open research questions. First, we focus on a particular type of financing contract and two types of capital market imperfections. The firm can also issue equity or raise external capital by other forms of loan contracts, or may be exposed to other capital market imperfections

such as taxes, agency costs etc. As the different capital market imperfections examined here show, the operational implications are expected to be model-specific.

Among these other market imperfections, agency costs arising from asymmetric information between the creditor and the firm are worth discussing. In our model, we assume that the creditor has perfect information about the firm. In reality, the creditor may not have perfect information about the risk profile of the operational investments, nor be able to monitor the firm after the loan is taken, nor have the same valuation of the collateralized assets as the firm. Each of these would create agency costs and impose additional financing frictions. Our analysis provides partial answers for this case. For example, if there is asymmetric information, and if there is no signalling by the firm or screening by the creditor, the creditor would offer identical financing costs for each technology with a secured loan. In this case, as we discussed in §5, the technology choice in imperfect capital markets is identical to the technology choice in perfect capital markets. With a proper modeling of the interaction between the creditor and the firm under asymmetric information, new trade-offs and new implications will arise as discussed in, for example, Brunet and Babich (2009).

Relaxing the assumptions we made on the production environment gives rise to a number of interesting possibilities, both in the theory of capacity management and integrated risk management. For example, we assume that second stage production is costless. With a positive production cost, the optimal production decision is limited by the cash availability of the firm (financial capacity constraint) in addition to the physical capacity constraint. This brings an additional facet to the problem that hampers tractability: The allocation of the financial capacity between the two stages; and between the products in the second stage in a two-product setting. We also assume that the internal budget of the firm is deterministic. This budget may depend on some economic factors and can be random. Moreover, if the internal budget depends on a tradable asset, then the firm can engage in financial risk management to engineer the budget as discussed in Froot et al (1993). The optimal technology choice (flexible versus dedicated) together with the decision of engaging in financial risk management form the optimal integrated risk management portfolio of the firm. Boyabatlı et al. (2011) analyze the effect of budget variability and financial risk management on the stochastic capacity investment problem with a more detailed model of the firm's production environment (that includes positive production cost and engaging in financial risk management) with hard financial constraints (no borrowing). It would be interesting to analyze the impact of endogenous credit terms on the integrated risk management portfolio of the firm (that consists of financial risk management and flexible versus dedicated technology choice).

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TECHNICAL APPENDIX TO:
STOCHASTIC CAPACITY INVESTMENT AND FLEXIBLE
VERSUS DEDICATED TECHNOLOGY CHOICE IN IMPERFECT
CAPITAL MARKETS

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§A contains the proofs for our technical statements in the paper. We present the proofs for technical statements that we develop in this Appendix in §B. We use the following identities for the standard normal random variable with cdf $\Phi(\cdot)$ and pdf $\phi(\cdot)$ throughout the Appendix: $\phi'(z) = -z\phi(z)$, $\int_{-\infty}^v z\phi(z)dz = -\phi(v)$ and $1 > \left[\frac{\phi(v)}{1-\Phi(v)}\right]^2 - \frac{v\phi(v)}{1-\Phi(v)} > 0$, where the last two inequalities are proven in Sampford (1953).

A Main Proofs

Proof of Proposition 1: If the firm does not have the limited liability option, then $\pi_D(K_D)$ is strictly concave in K_D and the unique optimal capacity investment level K_D^* and the optimal expected equity value π_D^* are given by

$$K_D^* = \begin{cases} K_D^0 \doteq \left(\frac{\bar{\xi}(1+\frac{1}{b})}{(1-\gamma_D)c_D}\right)^{-b} & \text{if } B \geq 2c_D K_D^0 \\ \frac{B}{2c_D} & \text{if } 2c_D K_D^1 \leq B < 2c_D K_D^0 \\ K_D^1 \doteq \left(\frac{\bar{\xi}(1+\frac{1}{b})}{(1+a_D-\gamma_D)c_D}\right)^{-b} & \text{if } B < 2c_D K_D^1, \end{cases} \quad (5)$$

$$\pi_D^* = \begin{cases} \frac{2c_D K_D^0(1-\gamma_D)}{-(b+1)} + B + P & \text{if } B \geq 2c_D K_D^0 \\ 2\bar{\xi}\frac{B}{2c_D} + \gamma_D B + P & \text{if } 2c_D K_D^1 \leq B < 2c_D K_D^0 \\ \frac{2c_D K_D^1(1+a_D-\gamma_D)}{-(b+1)} + B(1+a_D) + P & \text{if } B < 2c_D K_D^1 \end{cases}$$

With limited liability, when the firm borrows ($K_D > \frac{B}{2c_D}$), we have $l_D(K_D) \doteq K_D^{-\frac{1}{b}}(1+a_D-\gamma_D)2c_D - K_D^{(-1-\frac{1}{b})}[B(1+a_D)+P]$ such that the firm is able to pay back the face value of the loan

if and only if $\tilde{\xi}_1 + \tilde{\xi}_2$ is no less than $l_D(K_D)$. For $\tilde{\xi}_1 + \tilde{\xi}_2 > l_D(K_D)$, the optimal equity value $\Pi_D^* > 0$, and for $\tilde{\xi}_1 + \tilde{\xi}_2 \leq l_D(K_D)$, $\Pi_D^* = 0$. We obtain

$$\frac{\partial l_D(K_D)}{\partial K_D} = -\frac{1}{b} K_D^{\left(\frac{-1}{b}-1\right)} (1 + a_D - \gamma_D) 2c_D + \left(1 + \frac{1}{b}\right) K_D^{\left(-2-\frac{1}{b}\right)} [B(1 + a_D) + P] > 0. \quad (6)$$

Therefore, we can identify the unique $K_D^l < K_D^u$ such that $l_D(K_D^l) \doteq 2\xi^l$ and $l_D(K_D^u) \doteq 2\xi^u$. Since $l_D(K_D)$ is strictly increasing in K_D , we have $l_D(K_D) \geq 2\xi^u$ for $K_D \geq K_D^u$; hence $\Pi_D^* = 0$ at each $\tilde{\xi}_1 + \tilde{\xi}_2$ and $\pi_D^* = 0$ for $K_D \in [K_D^u, \infty)$. Therefore, it is sufficient to analyze the problem for $K_D \in [0, K_D^u)$. We have three separate cases to consider:

Case 1: For $K_D \in \left[0, \frac{B}{2c_D}\right]$, similar to the no limited liability case, the firm does not borrow, and the expected equity value of the firm is $\pi_D^* = \max_{K_D} 2\bar{\xi} K_D^{\left(1+\frac{1}{b}\right)} + B + P - 2c_D(1 - \gamma_D) K_D$.

Case 2: For $K_D \in \left[\frac{B}{2c_D}, K_D^l\right]$, similar to the no limited liability case, the firm optimally borrows, and is always able to pay back the face value of the loan.¹ The expected equity value of the firm is $\pi_D^* = \max_{K_D} 2\bar{\xi} K_D^{\left(1+\frac{1}{b}\right)} + B(1 + a_D) + P - 2c_D(1 + a_D - \gamma_D) K_D$.

Case 3: For $K_D \in (K_D^l, K_D^u)$ the firm always borrows, and for some demand realization, is not able to pay back the face value of the loan; hence the expected equity value of the firm is

$$\pi_D^* = \max_{K_D} \int \int_{\Upsilon_D(\boldsymbol{\xi}; K_D)} \left[(\tilde{\xi}_1 + \tilde{\xi}_2) K_D^{\left(1+\frac{1}{b}\right)} - 2c_D K_D (1 + a_D - \gamma_D) + B(1 + a_D) + P \right] f(\xi_1, \xi_2) d\xi_1 d\xi_2,$$

where $\Upsilon_D(\boldsymbol{\xi}; K_D) \doteq \{\boldsymbol{\xi} : \xi_1 + \xi_2 \geq l_D(K_D)\}$ and $f(\xi_1, \xi_2)$ is the joint pdf of $\boldsymbol{\xi}$.

Let $g_D(K_D)$ denote the objective function in the overall optimization problem and $g_D^i(K_D)$ denote the objective function in case i . It is easy to establish that $g_D(K_D)$ is continuous at the boundaries $K_D = \frac{B}{2c_D}$ and $K_D = K_D^l$; and hence $g_D(K_D)$ is continuous in K_D . It follows from (5) that $g_D(K_D)$ is strictly concave in K_D for $K_D \in [0, K_D^l]$ and has a kink at $K_D = \frac{B}{2c_D}$. We obtain

$$\frac{\partial g_D^3(K_D)}{\partial K_D} = \int \int_{\Upsilon_D(\boldsymbol{\xi}; K_D)} \left[\left(1 + \frac{1}{b}\right) (\xi_1 + \xi_2) K_D^{\left(\frac{1}{b}\right)} - 2(1 + a_D - \gamma_D) c_D \right] f(\xi_1, \xi_2) d\xi_1 d\xi_2. \quad (7)$$

It is easy to verify that $\frac{\partial g_D^3(K_D)}{\partial K_D} \Big|_{K_D^l-} = \frac{\partial g_D^3(K_D)}{\partial K_D} \Big|_{K_D^l+}$; hence $g_D(K_D)$ does not have a kink at $K_D = K_D^l$. Define $G(K_D, \boldsymbol{\xi}) \doteq \left(1 + \frac{1}{b}\right) (\xi_1 + \xi_2) K_D^{\left(\frac{1}{b}\right)} - 2(1 + a_D - \gamma_D) c_D$ as the integrand of (7) (without the density function). Note that $G(K_D, \boldsymbol{\xi})$ is increasing in

¹It can be shown that for $\xi^l \geq 0$ and $\gamma_D \geq 0$, $K_D^l \geq \frac{B}{2c_D}$, where the equality only holds if $\xi^l = 0$ and $\gamma_D = 0$.

ξ_i for $i = 1, 2$, and decreasing in K_D . We define $\hat{K}_D \doteq \left(\frac{\xi^u(1+\frac{1}{b})}{(1+a_D-\gamma_D)c_D} \right)^{-b}$. We have $l_D(\hat{K}_D) = 2(1+\frac{1}{b})\xi^u \left[1 - \frac{B(1+a_D)+P}{2\hat{K}_D(1+a_D-\gamma_D)c_D} \right] < 2\xi^u$, thus $\hat{K}_D < K_D^u$ and is in the feasible region of K_D . Note that for $\xi^{u'} \doteq (\xi^u, \xi^u)$, $G_D(\hat{K}_D, \xi^u) = 0$. Since $(\xi_1 + \xi_2)$ takes its maximum value at $\xi = \xi^u$, and $G_D(K_D, \xi)$ is strictly increasing in ξ_i for $i \in \{1, 2\}$, we have $G_D(K_D, \xi) < 0$ for $\xi \in \Upsilon_D(\xi, \hat{K}_D)$. Therefore $\frac{\partial g_D^3(K_D)}{\partial K_D} \Big|_{\hat{K}_D} < 0$. Since $G_D(K_D, \xi)$ is strictly decreasing in K_D , $\frac{\partial g_D^3(K_D)}{\partial K_D} < 0$ for $K_D \in [\hat{K}_D, K_D^u)$.

In summary, $g_D(K_D)$ is strictly concave in K_D for $K_D \in [0, K_D^l]$ (with a kink at $K_D = \frac{B}{2c_D}$), and is strictly decreasing in K_D for $K_D \in [\hat{K}_D, K_D^u)$. It follows that $g_D(K_D)$ will be unimodal if $K_D^l \geq \hat{K}_D$. Since $\frac{\partial l_D(K_D)}{\partial K_D} > 0$ (from (6)), this is equivalent to $l_D(\hat{K}_D) \leq 2\xi^l$, which gives us $B \geq B_D^h$. In this case, K_D^* is in the strictly concave part and is unique. K_D^* is identical to (5). ■

Proof of Proposition 2: In the proof of Proposition 1, we already established that the stage-1 objective function $g_D(K_D)$ is strictly concave in K_D for $K_D \in [0, K_D^l]$ and strictly decreasing in K_D for $K_D \in [\hat{K}_D, K_D^u)$. We obtain $\frac{\partial g_D(K_D)}{\partial K_D} \Big|_{K_D^l} = \frac{(1+\frac{1}{b})2\bar{\xi}}{(K_D^l)^{-1/b}} \left[1 - \left(\frac{K_D^l}{K_D^1} \right)^{-\frac{1}{b}} \right]$ where $K_D^1 = \left(\frac{\bar{\xi}(1+\frac{1}{b})}{(1+a_D-\gamma_D)c_D} \right)^{-b}$. It follows that $\frac{\partial g_D(K_D)}{\partial K_D} \Big|_{K_D^l} > 0$ if and only if $K_D^l < K_D^1$. In this case $g_D(K_D)$ is strictly increasing for $K_D \in [0, K_D^l]$ and strictly decreasing in K_D for $K_D \in [\hat{K}_D, K_D^u)$. Since $g_D(K_D)$ is continuous in K_D , there exists at least one $K_D^* \in (K_D^l, K_D^u)$ such that $\frac{\partial g_D(K_D)}{\partial K_D} \Big|_{K_D^*} = 0$. $MP_D(K_D)$ characterizes this first-order condition. Since $\frac{\partial l_D(K_D)}{\partial K_D} > 0$ (from (6)), $K_D^l < K_D^1$ is equivalent to $l_D(K_D^1) > 2\xi^l$, which gives $B < B_D^l$.

To prove that $K_D^* \in (K_D^l, K_D^u)$, it is sufficient to show that $\frac{\partial g_D(K_D)}{\partial K_D} > 0$ for $K_D \in (K_D^l, K_D^1]$. For $K_D > K_D^l$, as follows from (7), we have

$$\frac{\partial g_D(K_D)}{\partial K_D} = \left(1 + \frac{1}{b} \right) K_D^{\frac{1}{b}} \int \int_{\Upsilon_D(\xi; K_D)} \left[\xi_1 + \xi_2 - 2\bar{\xi} \left(\frac{K}{K^1} \right)^{-\frac{1}{b}} \right] f(\xi_1, \xi_2) d\xi_1 d\xi_2$$

Let $H_D(K_D) \doteq \int \int_{\Upsilon_D(\xi; K_D)} \left[\xi_1 + \xi_2 - 2\bar{\xi} \left(\frac{K}{K^1} \right)^{-\frac{1}{b}} \right] f(\xi_1, \xi_2) d\xi_1 d\xi_2$. Note that, for $K_D > K_D^l$, $H_D(K_D)$ and $\frac{\partial g_D(K_D)}{\partial K_D}$ have the same sign, so we can use $H_D(K_D)$ to characterize the sign of $\frac{\partial g_D^3(K_D)}{\partial K_D}$. Define $M_D(K_D, \xi) \doteq \xi_1 + \xi_2 - 2\bar{\xi} \left(\frac{K}{K^1} \right)^{-\frac{1}{b}}$ as the integrand in $H_D(K_D)$ (without the density function). For ξ such that $\xi_1 + \xi_2 = l_D(K_D)$, we obtain $M_D(K_D, \xi) = K_D^{-\frac{1}{b}} \left[\frac{2(1+a_D-\gamma_D)c_D}{(b+1)} - \frac{B(1+a_D)+P}{K_D} \right] < 0$ since $b < -1$. As $M_D(K_D, \xi)$ is strictly increasing in ξ_i for $i \in \{1, 2\}$, $M_D(K_D, \xi) < 0$ for ξ such that $\xi_1 + \xi_2 < l_D(K_D)$. Therefore, we have $H_D(K_D) > \int \int \left[\xi_1 + \xi_2 - 2\bar{\xi} \left(\frac{K_D}{K_D^1} \right)^{-\frac{1}{b}} \right] f(\xi_1, \xi_2) d\xi_1 d\xi_2 = 2\bar{\xi} \left[1 - \left(\frac{K_D}{K_D^1} \right)^{-\frac{1}{b}} \right]$. For $K_D \leq$

K_D^1 , we have $2\bar{\xi} \left[1 - \left(\frac{K_D}{K_D^1} \right)^{-\frac{1}{b}} \right] \geq 0$; and hence $H_D(K_D) > 0$ for $K_D \in (K_D^l, K_D^1]$. ■

Proof of Proposition 3: The stage-1 objective function $g_D(K_D)$ is strictly concave in K_D for $K_D \in [0, K_D^l]$ (and strictly increasing at $K_D = 0$) and strictly decreasing in K_D for $K_D \in [\hat{K}_D, K_D^u]$. If π_D is unimodal in K_D , thus $g_D(K_D)$ is unimodal in K_D , it follows that

1. If $\frac{\partial g_D^2(K_D)}{\partial K_D} \Big|_{K_D^l} \leq 0$, then the unique K_D^* is characterized by the strictly concave part (similar to Proposition 1).
2. If $\frac{\partial g_D^2(K_D)}{\partial K_D} \Big|_{K_D^l} > 0$, then the unique K_D^* is characterized by $MP_D(K_D^*) = 0$ as defined in Proposition 2. Let \bar{K}_D denote the optimal solution in this case. From Proposition 2, we have $\bar{K}_D \geq K_D^1$.

As it follows from the proof of Proposition 2, $\frac{\partial g_D^2(K_D)}{\partial K_D} \Big|_{K_D^l} > 0$ is equivalent to $B < B_D^l$. The optimal expected equity value of the firm, π_D^* , follows directly.

We now prove that π_D^* decreases in a_D . We have two cases to consider:

Case 1: $2c_D K_D^1 \left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})} \right] \left[1 - \frac{\gamma_D}{1+a_D} \right] - \frac{P}{1+a_D} \leq B < 2c_D K_D^1$

The firm's optimal expected equity value is given by $\pi_D^* = \frac{2(1+a_D-\gamma_D)c_D K_D^1(a_D)}{-(b+1)} + B(1+a_D) + P$ where $K_D^1(a_D) = \left(\frac{\bar{\xi}(1+\frac{1}{b})}{(1+a_D-\gamma_D)c_D} \right)^{-b}$. We obtain $\frac{\partial \pi_D^*}{\partial a_D} = -2c_D K_D^1(a_D) + B < 0$ as follows from the definition of Case 1.

Case 2: $0 \leq B < 2c_D K_D^1 \left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})} \right] \left[1 - \frac{\gamma_D}{1+a_D} \right] - \frac{P}{1+a_D}$

The firm's optimal expected equity value is given by

$$\pi_D^* = \int \int_{\Upsilon_D(\xi; \bar{K}_D)} \left[(\tilde{\xi}_1 + \tilde{\xi}_2) \bar{K}_D^{(1+\frac{1}{b})} - 2c_D \bar{K}_D (1+a_D - \gamma_D) + B(1+a_D) + P \right] f(\xi_1, \xi_2) d\xi_1 d\xi_2.$$

Note that $\frac{\partial \pi_D^*(\bar{K}_D)}{\partial a_D} = \frac{\partial \pi_D(K_D)}{\partial a_D} \Big|_{\bar{K}_D} + \frac{\partial \pi_D(K_D)}{\partial K_D} \Big|_{\bar{K}_D} \frac{\partial \bar{K}_D}{\partial a_D}$. Since $\frac{\partial \pi_D(K_D)}{\partial K_D} \Big|_{\bar{K}_D} = 0$, we obtain

$$\frac{\partial \pi_D(K_D)}{\partial a_D} \Big|_{\bar{K}_D} = \int \int_{\Upsilon_D(\xi; \bar{K}_D)} [-2c_D \bar{K}_D + B] f(\xi_1, \xi_2) d\xi_1 d\xi_2 < 0$$

as follows from $K_D^1 < \bar{K}_D$ and the definition of Case 2. ■

Lemma A.1 *If $b \geq -2$ and ξ has a bivariate normal distribution, then π_D is unimodal in K_D .*

Proof of Proposition 4: We define $S^1(a_D) \doteq 2c_D K_D^0 (1 - \gamma_D)^{-b} \left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})} \right] \frac{(1+a_D-\gamma_D)^{(b+1)}}{(1+a_D)}$

such that for a given a_D , for $B \geq S^1(a_D)$, the firm uses a secured loan (and invests in $K_D^*(a_D) = K_D^1(a_D)$) without default possibility. $B \geq S^1(a_D)$ is equivalent to $d_D(K_D^1) \leq 2\xi^l$. Hence, both the default cost and the expected loss due to the unsecured part of the loan

are 0 in (4). We define $S^2(a_D) \doteq S^1(a_D) - \frac{P}{(1+a_D)}$ such that for $S^1(a_D) > B \geq S^2(a_D)$, the firm uses a secured loan (and invests in $K_D^*(a_D) = K_D^1(a_D)$) with default possibility. $B \geq S^2(a_D)$ is equivalent to $l_D(K_D^1) \leq 2\xi^l$. Hence, the default cost is strictly positive but the expected loss due to the unsecured part of the loan is 0 in (4). For $B < S^2(a_D)$, the firm optimally borrows to invest in $K_D^*(a) = \bar{K}_D(a_D)$. In this case, the firm uses an unsecured loan and both the default cost and the expected loss due to the unsecured part of the loan are strictly positive in (4).

In summary, for any given a_D , the ordering of B and thresholds $S^1(a_D)$ and $S^2(a_D)$ determine the optimal borrowing level of the firm, and hence the form of $\Lambda_D(a_D)$. We obtain $\frac{\partial S^1(a_D)}{\partial a_D} < 0$ for $a_D \in [0, a_D^{max})$ thus, we can analyze the problem in two cases.

Case 1: $B \geq S^1(0) = 2c_D K_D^0(1 - \gamma_D) \left[1 - \frac{\xi^l}{\xi(1+\frac{1}{b})} \right]$.

As $S^1(a_D)$ is strictly decreasing, we have $B \geq S^1(a_D)$ (and hence $B > S^2(a_D)$) $\forall a_D \in [0, a_D^{max})$. Therefore, we have $\Lambda_D(a_D) = (2c_D K_D^1(a_D) - B)a_D$ for $0 \leq a_D < a_D^{max}$.

Case 2: $B < S^1(0) = 2c_D K_D^0(1 - \gamma_D) \left[1 - \frac{\xi^l}{\xi(1+\frac{1}{b})} \right]$.

In this case, the ordering of B and $S^2(a_D)$ is important in characterizing $\Lambda_D(a_D)$. We have $S^2(0) = 2c_D K_D^0(1 - \gamma_D) \left[1 - \frac{\xi^l}{\xi(1+\frac{1}{b})} \right] - P$ and we obtain

$$\frac{\partial S^2(a_D)}{\partial a_D} = \frac{1}{(1+a_D)^2} \left[P - 2c_D K_D^0(1 - \gamma_D)^{-b} \left(1 - \frac{\xi^l}{\xi(1+\frac{1}{b})} \right) (1+a_D - \gamma_D)^b [-b(1+a_D) - \gamma_D] \right]$$

Notice that $S^2(0)$ is positive (negative) if P is less (greater) than $2c_D K_D^0(1 - \gamma_D) \left[1 - \frac{\xi^l}{\xi(1+\frac{1}{b})} \right]$. Since $(1+a_D - \gamma_D)^b [-b(1+a_D) - \gamma_D]$ is strictly decreasing in a_D , for $P \geq 2c_D K_D^0(-b - \gamma_D) \left[1 - \frac{\xi^l}{\xi(1+\frac{1}{b})} \right]$, we have $\frac{\partial S^2(a_D)}{\partial a_D} \geq 0$ for $a_D \geq 0$. For $P < 2c_D K_D^0(-b - \gamma_D) \left[1 - \frac{\xi^l}{\xi(1+\frac{1}{b})} \right]$, there exists a unique \underline{a}_D such that $\frac{\partial S^2(a_D)}{\partial a_D} \leq 0$ for $a_D \leq \underline{a}_D$ and $\frac{\partial S^2(a_D)}{\partial a_D} > 0$ for $a_D > \underline{a}_D$. Since the signs of $S^2(0)$ and $\frac{\partial S^2(a_D)}{\partial a_D}$ depend on P , we have three subcases. Before analyzing them, we first present a Lemma that we will use throughout the rest of the proof.

Lemma A.2 *We have $B \geq S^1(a_D^{max}) > S^2(a_D^{max})$, $\forall B \geq 0$.*

Subcase 2.1: $P \geq 2c_D K_D^0 \left[1 - \frac{\xi^l}{\xi(1+\frac{1}{b})} \right] (-b - \gamma_D)$.

In this case, we have $S^2(0) < 0$ and $\frac{\partial S^2(a_D)}{\partial a_D} \geq 0$, $\forall a_D$. For $a_D^{max} = \left[\left(\frac{2c_D K_D^0}{B} \right)^{-\frac{1}{b}} - 1 \right] (1 - \gamma_D)$, we obtain $S^2(a_D^{max}) < 0$. Hence, for $a_D \in [0, a_D^{max})$, we have $S^2(a_D) < 0 < B$. It follows that the firm always uses a secured loan (and invests in $K_D^1(a_D)$). For $B < S^1(0)$ (which follows from the definition of Case 2), since $S^1(a_D)$ is strictly decreasing in a_D and $B \geq S^1(a_D^{max})$ (from Lemma A.2), it follows that there exists a unique a_D^d , as defined by $S^1(a_D^d) \doteq B$

(where the superscript d refers to “default”). We have $B < S^1(a_D)$ for $a_D < a_D^d$, and the firm uses a secured loan with default possibility, and $B \geq S^1(a_D)$ for $a_D \geq a_D^d$, the firm uses a secured loan without default possibility. Therefore, $\Lambda_D(a_D)$ is characterized by

$$\Lambda_D(a_D) = \begin{cases} (2c_D K_D^1(a_D) - B)a_D - F(d_D(K_D^1(a_D)))S & \text{if } 0 \leq a_D < a_D^d \\ (2c_D K_D^1(a_D) - B)a_D & \text{if } a_D^d \leq a_D < a_D^{max}. \end{cases},$$

Subcase 2.2: $2c_D K_D^0 \left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})} \right] (1 - \gamma_D) \leq P < 2c_D K_D^0 \left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})} \right] (-b - \gamma_D)$.

We have $S^2(0) \leq 0$, and $S^2(a_D)$ is first strictly decreasing, and then strictly increasing in a_D . We obtain $S^2(a_D^{max}) < 0$; hence $S^2(a_D) < 0$ for $a_D \in [0, a_D^{max})$ in this case. Therefore $\Lambda_D(a_D)$ is identical to subcase 2.1.

Subcase 2.3: $2c_D K_D^0 \left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})} \right] (1 - \gamma_D) > P$

We have $S^2(0) > 0$, and $S^2(a_D)$ is first strictly decreasing, and then strictly increasing in a_D .

If $B \geq S^2(0)$ (and $B < S^1(0)$ by definition of Case 2), since $B \geq S^1(a_D^{max}) > S^2(a_D^{max})$ (from Lemma A.2), $\Lambda_D(a_D)$ is characterized in a similar fashion to the other two subcases. If $B < S^2(0)$, as $S^2(a_D)$ is first strictly decreasing, and then strictly increasing in a_D and $B \geq S^1(a_D^{max}) > S^2(a_D^{max})$ (from Lemma A.2), there exists a unique $a_D^l \in [0, a_D^{max})$, as defined in $S^2(a_D^l) \doteq B$ (where the superscript l refers to “limited liability”). We have $B < S^2(a_D)$ for $a_D < a_D^l$ and $B \geq S^2(a_D)$ for $a_D \geq a_D^l$. Since $S^2(a_D) = S^1(a_D) - \frac{P}{1+a_D}$, it follows that $a_D^l \leq a_D^d$, with equality only holding for $P = 0$. Therefore, we have the following three regions: For $a_D < a_D^l (< a_D^d)$, we have $B < S^2(a_D)$ (and $B < S^1(a_D)$), the firm uses an unsecured loan; for $a_D^l \leq a_D < a_D^d$, we have $S^2(a_D) \leq B < S^1(a_D)$, and the firm uses a secured loan with default possibility; and for $a_D \geq a_D^d$, we have $S^2(a_D) < S^1(a_D) \leq B$, and the firm uses a secured loan without default possibility. ■

Proof of Proposition 5: Since this equilibrium is relevant for firms that may default but use a secured loan (Case *ii* of Proposition 4) and firms that may use an unsecured loan (Case *iii* of Proposition 4); we will analyze these two cases separately. At equilibria where the firm uses a secured loan with default possibility, the creditor’s expected return with the dedicated technology is given by

$$\Lambda_D(a_D) = \left(2c_D \dot{K}_D^1 - B \right) \dot{a}_D - S \times Pr \left(\xi_1 + \xi_2 < d_D(\dot{K}_D^1) \right),$$

where $d_D(\dot{K}_D^1) = 2\bar{\xi} \left(1 + \frac{1}{b} \right) \left[1 - \frac{B(1+\dot{a}_D)}{2c_D \dot{K}_D^1 (1+\dot{a}_D - \gamma_D)} \right]$. Since ξ has a bivariate normal distribution, $\xi_1 + \xi_2$ is normally distributed with mean $\bar{\mu} = 2\bar{\xi}$ and standard deviation $\bar{\sigma} = \sigma\sqrt{2(1+\rho)}$. Since $b < -1$ and $B < 2c_D \dot{K}_D^1 [1 - \gamma_D]$, we obtain $d_D(\dot{K}_D^1) < \bar{\mu}$. We

have $Pr\left(\xi_1 + \xi_2 < d_D(\dot{K}_D^1)\right) = \Phi\left(\frac{d_D(\dot{K}_D^1) - \bar{\mu}}{\bar{\sigma}}\right)$ where $\Phi(\cdot)$ is the cdf of the standard normal random variable.

For firms that may default but use a secured loan (Case *ii* of Proposition 4), for any $a_D \in [0, a_D^d)$, we obtain

$$\begin{aligned}\frac{\partial \Lambda_D(a_D)}{\partial \rho} &= -BC \phi\left(\frac{d_D(K_D^1) - \bar{\mu}}{\bar{\sigma}}\right) \left(\frac{\bar{\mu} - d_D(K_D^1)}{\bar{\sigma}^2}\right) \frac{\partial \bar{\sigma}}{\partial \rho} < 0, \\ \frac{\partial \Lambda_D(a_D)}{\partial \sigma} &= -BC \phi\left(\frac{d_D(K_D^1) - \bar{\mu}}{\bar{\sigma}}\right) \left(\frac{\bar{\mu} - d_D(K_D^1)}{\bar{\sigma}^2}\right) \frac{\partial \bar{\sigma}}{\partial \sigma} < 0\end{aligned}$$

where $\phi(\cdot)$ is the density function of the standard normal random variable, as follows from $\frac{\partial}{\partial \rho} \bar{\sigma} = \frac{\sigma}{\bar{\sigma}} > 0$, $\frac{\partial}{\partial \sigma} \bar{\sigma} = \sqrt{2(1+\rho)} > 0$, and $d_D(K_D^1) < \bar{\mu}$. From the Pareto-optimality of the equilibrium, i.e. \dot{a}_D is the minimum a_D that satisfies $\Lambda_D(a_D) = 0$, it follows that with an increase in σ or ρ , \dot{a}_D increases.

For firms that may use an unsecured loan, since $\dot{a}_D \in [a_D^l, a_D^d)$, it follows from above that $\left.\frac{\partial}{\partial \tau} \Lambda_D(a_D)\right|_{\dot{a}_D} < 0$ for $\tau \in \{\sigma, \rho\}$. In fact, $\Lambda_D(a_D)$ is decreasing in σ or ρ for any $\dot{a}_D \in [a_D^l, a_D^d)$, but we cannot characterize the effect of σ or ρ on $\Lambda_D(a_D)$ for $a_D \in [0, a_D^l)$. Let $\dot{a}_D(\tau)$ denote the equilibrium financing cost for a given $\tau \in \{\sigma, \rho\}$. With a small increment in τ from τ_0 to τ_1 , we can guarantee that $\Lambda_D(a_D; \tau_1) < 0$ for $\forall a_D < \dot{a}_D(\tau_0)$ because i) $\Lambda_D(a_D; \tau_0) < 0$ for $\forall a_D < \dot{a}_D(\tau_0)$ from the definition of the equilibrium, and ii) $\left|\frac{\partial}{\partial \tau} \Lambda_D(a_D)\right|$ and $\left|\frac{\partial}{\partial a_D} \Lambda_D(a_D)\right|$ are bounded. Therefore \dot{a}_D increases (locally) in $\tau \in \{\sigma, \rho\}$.

Since for a given a_D , π_D^* is independent of $\tau \in \{\sigma, \rho\}$, we have $\left.\frac{\partial \pi_D^*}{\partial \tau}\right|_{\dot{a}_D} = \frac{\partial \pi_D^*}{\partial a_D} \left.\frac{\partial a_D}{\partial \tau}\right|_{\dot{a}_D}$. From Proposition 3, we have $\frac{\partial \pi_D^*}{\partial a_D} < 0$, hence $\frac{\partial \pi_D^*}{\partial \tau} < 0$. Similarly, $\dot{K}_D = K_D^1(\dot{a}_D)$ is independent of $\tau \in \{\sigma, \rho\}$; hence we have $\left.\frac{\partial \dot{K}_D}{\partial \tau}\right|_{\dot{a}_D} = \frac{\partial K_D^1(a_D)}{\partial a_D} \left.\frac{\partial a_D}{\partial \tau}\right|_{\dot{a}_D}$. Since $K_D^1(a_D)$ decreases in a_D , we have $\frac{\partial \dot{K}_D}{\partial \tau} < 0$ for $\tau \in \{\sigma, \rho\}$. ■

Lemma A.3 *If $b \geq -2$ and ξ has a bivariate normal distribution, then for a given financing cost a_D with the dedicated technology, for the firm that uses an unsecured loan, K_D^* and π_D^* increase in σ and ρ , and decrease in a_D .*

Lemma A.4 *If $b \geq -2$ and ξ has a bivariate normal distribution, when the firm uses an unsecured loan with the dedicated technology, the creditor's net gain from secured lending and its expected loss due to the unsecured part of the loan increase in σ and ρ . Its expected default cost increases in σ and ρ if $d_D(\bar{K}_D(a_D)) \leq 2\bar{\xi}$.*

Proof of Remark 2 The form of $\bar{c}_D^P(c_F)$ follows from a direct comparison of $\dot{\pi}_D$ and $\dot{\pi}_F$ in perfect capital markets. Since $\gamma_F \geq \gamma_D$ by assumption, to prove $\bar{c}_D^P(c_F) \leq c_F$ it is sufficient

to show $\mathbb{E}^{-b} \left[\left(\xi_1^{-b} + \xi_2^{-b} \right)^{-\frac{1}{b}} \right] \geq 2\bar{\xi}^{-b}$. From Hardy et. al (1988, p.146), if $d \in (0, 1)$ and X, Y are non-negative random variables then the following is true: $\mathbb{E}^{1/d} \left[(X + Y)^d \right] \geq \mathbb{E}^{1/d} [X^d] + \mathbb{E}^{1/d} [Y^d]$ where equality only holds when X and Y are effectively proportional, i.e. $X = \lambda Y$. In $\bar{c}_D^P(c_F)$, we have $d = -\frac{1}{b} \in (0, 1)$ and $\boldsymbol{\xi} \geq \boldsymbol{\xi}^l \geq \mathbf{0}$, replacing X with ξ_1^{-b} and Y with ξ_2^{-b} gives the desired result. Notice that $\bar{c}_D^P(c_F) = c_F$ only if $\xi_1 = \xi_2$ (since we focus on the symmetric bivariate distribution) and $\gamma_F = \gamma_D$. $\xi_1 = \xi_2$ is only possible if either $\boldsymbol{\xi}$ is deterministic or $\rho = 1$. ■

B Proofs for Supporting Lemmas

Proof of Lemma A.1: Since $\boldsymbol{\xi}$ has a bivariate normal distribution, $\psi \doteq \xi_1 + \xi_2$ is normally distributed with mean $\bar{\mu} = 2\bar{\xi}$ and standard deviation $\bar{\sigma} = \sigma\sqrt{2(1+\rho)}$. Let $F(\cdot)$ denote the cdf of ψ , and $\bar{F}(\cdot) = 1 - F(\cdot)$. By using ψ , as follows from the proof of Proposition 2; for $K_D \geq K_D^l$, we have $\text{sgn} \left(\frac{\partial g_D^3(K_D)}{\partial K_D} \right) = \text{sgn}(H_D(K_D))$ where $H_D(K_D) = \int_{l_D(K_D)}^{\xi^u} \left[\psi - 2\bar{\xi} \left(\frac{K_D}{K_D^l} \right)^{-\frac{1}{b}} \right] f(\psi) d\psi$. Therefore we will focus on $H_D(K_D)$ to prove the unimodality of $g_D(K_D)$. From integration by parts, we obtain

$$H_D(K_D) = \int_{l_D(K_D)}^{2\xi^u} \bar{F}(\psi) d\psi - \bar{F}(l_D(K_D)) \left[K_D^{-\frac{1}{b}} \left(\frac{2(1+a_D-\gamma_D)c_D}{-(b+1)} + \frac{B(1+a_D)+P}{K_D} \right) \right].$$

Define $\Delta(K_D) \doteq K_D^{-\frac{1}{b}} \left(\frac{2(1+a_D-\gamma_D)c_D}{-(b+1)} + \frac{B(1+a_D)+P}{K_D} \right)$. We obtain

$$\frac{\partial \Delta(K_D)}{\partial K_D} = \left(1 + \frac{1}{b} \right) K_D^{-1} \left(l_D(K_D) + \frac{-b(b+2)}{(b+1)^2} 2(1+a_D-\gamma_D)c_D K_D^{-\frac{1}{b}} \right).$$

Note that for $K_D > K_D^l$, $l_D(K_D) > 2\xi^l \geq 0$; hence for $b \geq -2$ the second term is positive and $\frac{\partial \Delta(K_D)}{\partial K_D} > 0$ for $K_D > K_D^l$. We obtain $H_D(K_D) = \bar{F}(l_D(K_D)) \left[\frac{\int_{l_D(K_D)}^{2\xi^u} \bar{F}(\psi) d\psi}{\bar{F}(l_D(K_D))} - \Delta(K_D) \right]$.

As $\Delta(K_D)$ is increasing in K_D , if we can show that $\frac{\int_{l_D(K_D)}^{2\xi^u} \bar{F}(\psi) d\psi}{\bar{F}(l_D(K_D))}$ is decreasing in K_D , then for $K_D > K_D^l$, $H_D(K_D)$ can only change sign once, which is from positive to negative.

We now show that $\frac{\int_{l_D(K_D)}^{2\xi^u} \bar{F}(\psi) d\psi}{\bar{F}(l_D(K_D))}$ is decreasing in K_D . Since ψ is normally distributed with mean $\bar{\mu}$ and standard deviation $\bar{\sigma}$, by using the standard normal random variable, this expression can be written as

$$\frac{-1}{\left[1 - \Phi \left(\frac{l_D(K_D) - \bar{\mu}}{\bar{\sigma}} \right) \right]^2} \frac{\partial l_D(K_D)}{\partial K_D} \left[\left[1 - \Phi \left(\frac{l_D(K_D) - \bar{\mu}}{\bar{\sigma}} \right) \right]^2 - \phi \left(\frac{l_D(K_D) - \bar{\mu}}{\bar{\sigma}} \right) \int_{\frac{l_D(K_D) - \bar{\mu}}{\bar{\sigma}}}^{\infty} (1 - \Phi(z)) dz \right] \quad (8)$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cdf and pdf of the standard normal random variable respectively. Since $\frac{\partial l_D(K_D)}{\partial K_D} > 0$, it is sufficient to show that the last term in parenthesis is positive. Let

$v = \frac{l_D(K_D) - \bar{\mu}}{\bar{\sigma}}$. Using integration by parts, we obtain $\int_v^\infty (1 - \Phi(z)) dz = \phi(v) - v(1 - \Phi(v))$. Substituting this in (8), it is sufficient to show that $1 > \left[\frac{\phi(v)}{1 - \Phi(v)} \right]^2 - \frac{v\phi(v)}{1 - \Phi(v)}$ which directly follows from Sampford (1953). ■

Proof of Lemma A.3: For a given a_D , the optimal expected equity value π_D^* is given by

$$\left[1 - \Phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right) \right] \left[\bar{\mu} \bar{K}_D^{(1+\frac{1}{b})} + B(1 + a_D) + P - 2(1 + a_D)c_D \bar{K}_D \right] + \bar{\sigma} \bar{K}_D^{(1+\frac{1}{b})} \phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right).$$

where $\bar{\mu} = 2\bar{\xi}$ and $\bar{\sigma} = \sigma\sqrt{2(1+\rho)}$. Since $\bar{\sigma}$ is increasing in σ or ρ , it is sufficient to analyze the impact of $\bar{\sigma}$. We have $\frac{\partial \pi_D^*}{\partial \bar{\sigma}} = \frac{\partial \pi_D}{\partial \bar{\sigma}} \Big|_{\bar{K}_D} + \frac{\partial \pi_D}{\partial \bar{\sigma}} \Big|_{\bar{K}_D}$ where the first term is zero from the optimality of \bar{K} . We obtain $\frac{\partial \pi_D}{\partial \bar{\sigma}} \Big|_{\bar{K}_D} = \bar{K}_D^{(1+\frac{1}{b})} \phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right) > 0$.

$$\text{For } K_D^* = \bar{K}_D, \text{ since } \bar{K}_D \text{ is the unique maximizer, we have } \text{sgn} \left(\frac{\partial \bar{K}_D}{\partial \bar{\sigma}} \right) = \text{sgn} \left(\frac{\partial MP_D(K_D)}{\partial \bar{\sigma}} \Big|_{\bar{K}_D} \right).$$

Using the optimality condition

$$\left[1 - \Phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right) \right] \left[\left(1 + \frac{1}{b}\right) \bar{\mu} \bar{K}_D^{\frac{1}{b}} - 2(1 + a_D)c_D \right] = -(1 + \frac{1}{b}) \bar{\sigma} \bar{K}_D^{\frac{1}{b}} \phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right), \quad (9)$$

$$\text{we obtain } \frac{\partial MP_D(K_D)}{\partial \bar{\sigma}} \Big|_{\bar{K}_D} = (1 + \frac{1}{b}) \bar{K}_D^{\frac{1}{b}} \phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right) \left[\left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right)^2 + 1 - \frac{\phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right) \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right)}{1 - \Phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right)} \right].$$

Let $z = \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right)$. We need to show that $1 > z \left[\frac{\phi(z)}{1 - \Phi(z)} - z \right]$. It follows from Sampford (1953) that $\left[\frac{\phi(z)}{1 - \Phi(z)} - z \right] < \frac{1 - \Phi(z)}{\phi(z)}$; therefore it is sufficient to show $1 > \frac{z(1 - \Phi(z))}{\phi(z)}$ which also follows from Sampford (1953).

For the impact of a_D on \bar{K}_D , we have $\text{sgn} \left(\frac{\partial \bar{K}_D}{\partial a_D} \right) = \text{sgn} \left(\frac{\partial MP_D(K_D)}{\partial a_D} \Big|_{\bar{K}_D} \right)$. Using the optimality condition in (9), we obtain $\frac{\partial MP_D(K_D)}{\partial a_D} \Big|_{\bar{K}_D} =$

$$\left[1 - \Phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right) \right] \times \left[-2c_D + \left(1 + \frac{1}{b}\right) \bar{K}_D^{\frac{1}{b}} \frac{\partial l_D(\bar{K}_D)}{\partial a_D} \left[\left(\frac{\phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right)}{1 - \Phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right)} \right)^2 - \frac{\phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right) \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right)}{1 - \Phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right)} \right] \right].$$

Denoting Y as the last expression in brackets and using $\frac{\partial l_D(\bar{K}_D)}{\partial a_D} = K_D^{-\frac{1}{b}} \left[2c_D - \frac{B}{K_D} \right]$, the desired result follows because $-2c_D + \left(1 + \frac{1}{b}\right) \left[2c_D - \frac{B}{K_D} \right] Y < 0$ as $Y < 1$ from Sampford (1953). ■

Proof of Lemma A.4: We only provide the proof for the expected loss due to the unsecured part of the loan. The proofs for the default risk and the net gain from secured lending can be obtained in a similar fashion, and are omitted. Since $\bar{\sigma}$ is increasing in σ or ρ , it is sufficient to analyze the impact of $\bar{\sigma}$. By using the standard normal random variable,

the expected loss due to the unsecured part of the loan can be written as

$$\Phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right) \left[\bar{K}_D(1 + a_D)2c_D - \bar{\mu}\bar{K}_D^{(1+\frac{1}{b})} - B(1 + a_D) - P \right] + \bar{\sigma}\phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right) \bar{K}_D^{(1+\frac{1}{b})}.$$

Taking the derivative with respect to $\bar{\sigma}$, and using the optimality condition in (9), the derivative with respect to $\bar{\sigma}$ is given by $\phi \left(\frac{l_D(\bar{K}_D) - \bar{\mu}}{\bar{\sigma}} \right) \bar{K}_D^{(1+\frac{1}{b})} + \frac{\partial \bar{K}_D}{\partial \bar{\sigma}} \left[2(1 + a_D)c_D - \bar{\mu}\bar{K}_D^{\frac{1}{b}} \right]$.

This term is positive because $\frac{\partial \bar{K}_D}{\partial \bar{\sigma}} > 0$ from Lemma A.3 and the last expression is positive from the optimality condition in (9). ■

C References

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