Inventory Management with Advance Demand Information and Flexible Delivery

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This paper considers inventory models with advance demand information and flexible delivery. Customers place their orders in advance, and delivery is flexible in the sense that early shipment is allowed. Specifically, an order placed at time $t$ by a customer with demand leadtime $T$ should be fulfilled by period $t+T$; failure to fulfill it within the time window $[t, t+T]$ is penalized. We consider two situations: (1) customer demand leadtimes are homogeneous and demand arriving in period $t$ is a scalar $d_t$ to be satisfied within $T$ periods. We show that state-dependent $(s, S)$ policies are optimal, where the state represents advance demands outside the supply leadtime horizon. We find that increasing the demand leadtime is more beneficial than decreasing the supply leadtime. (2) Customers are heterogeneous in their demand leadtimes. In this case, demands are vectors and may exhibit crossover, necessitating an allocation decision in addition to the ordering decision. We develop a lower-bound approximation based on an allocation assumption, and propose protection level heuristics that yield upper bounds on the optimal cost. Numerical analysis quantifies the optimality gaps of the heuristics (2% on average for the best heuristic) and the benefit of delivery flexibility (14% on average using the best heuristic), and provides insights into when the heuristics perform the best and when flexibility is most beneficial.

Key words: Stochastic Inventory Model; Advance Demand Information; Flexible Delivery

1. Introduction

Order An Introduction to Probability Theory and Its Applications by Feller from Amazon, and you will be promised that the book will ship within 14 days. Order a popular item such as the Apple iPod Nano and Eminem’s CD, and you will be told that your purchase usually ships within 24 hours. In addition to the standard delivery option (e.g. 14 days for Feller’s book), there are options like “Guaranteed Accelerated 1-day Delivery” and “Guaranteed Accelerated 2-day Delivery” at different shipping costs. Shipping fees are guaranteed to be refunded if items fail to arrive on or
Flexible delivery arrangements where the customer accepts delivery of the product at any time before its quoted due date are prevalent in many instances where companies sell directly to consumers. In contrast, the inventory management literature on advance demand information (ADI) focuses almost exclusively on exact delivery: When customers place an order with a given due date, it is assumed that they will not accept early delivery. In other words, early shipment is forbidden. At the same time, delayed shipment is penalized. Hariharan and Zipkin (1995), one of the first papers to incorporate advance demand information into inventory management, provides the following justification: “This assumption is realistic in many though not all situations. The costs to customers of early deliveries are now widely appreciated, partly due to the JIT movement.” The authors argue that early payment associated with early delivery is a deterrent. The additional inventory cost borne by the customer, and the uncertainty in delivery timing may also make flexible delivery unappealing to the customer.

Much of the literature following has taken the exact delivery assumption for granted. However, in many situations where companies interact with end users directly (e.g. online retailing, services), it is customary for firms to tell their customers that the product/service will be delivered by a particular due date. It is easy to see why this is acceptable: If the product is for use or consumption, customers would typically prefer receiving their goods earlier rather than later. In this case, early delivery offers firms a powerful mechanism to reduce their inventory costs by transforming the firm’s inventory cost into the customers’ utility. Some firms recognize the variety in customer preferences and offer a range of options. For example, Dell’s Intelligent Fulfillment program includes both delivery within five days and delivery on an exact date in its delivery options (Özer and Wei 2004).

In this paper, we analyze inventory management with advance demand information and the possibility of early shipment, which we call flexible delivery. The model we use is closely related to the discrete-time, uncapacitated, advance demand information model of Gallego and Özer (2001), except that we allow for delivery flexibility. We first consider a model where customers are homogeneous in that they all have an identical demand leadtime $T$: Demand $d_i$ observed in period $i$ needs
to be satisfied on or before period $i + T$. The supply leadtime is $L$. Flexible delivery introduces a nonlinearity into the system evolution equations. Nevertheless, we show that the structure of the optimal solution parallels that of Gallego and Özer (2001): If $T \leq L + 1$, the system reduces to the traditional model by replacing the inventory position with the modified inventory position, and a modified $(s, S)$ policy is optimal; if $T > L + 1$, a state-dependent $(s(\hat{V}), S(\hat{V}))$ policy is optimal, where the state $\hat{V}$ represents information about advance demands beyond the supply leadtime.

We next turn to the more general model where customers are heterogeneous in their demand leadtimes: There are $T + 1$ categories of customers, with demand leadtimes ranging from 0 to $T$. Demand in period $i$ is now a vector $(d^i_0, d^i_1, \ldots, d^i_T)$, where $d^i_j$ stands for orders received in period $i$ and to be satisfied by the end of period $j$. Unlike the homogeneous demand case, it is no longer optimal to satisfy orders as early as possible since future orders may be due earlier (called “demand cross-over”) with $T \geq 2$. Fulfilling observed advance orders early reduces holding cost, but at the same time, increases the probability of shortage as unobserved urgent orders may arrive in the future. Besides choosing when and how much to order (ordering decision), now inventory managers have to decide when and by how much to fulfill advance orders (allocation decision). As the analysis becomes intractable in this case, we develop an approximation that relaxes the nonnegativity constraints on delivery quantities. This is equivalent to allowing the firm to take previously “mis-allocated” units back and to reuse them to satisfy urgent demands. Imposing such a relaxation helps bypass the allocation decisions, and ensures that myopic allocation is optimal for the relaxed problem. This approximation yields a lower bound on the optimal objective function value. We then propose three protection level heuristics (PL(0), PL(σ), and PL(Σ)) that use different levels (“zero,” “optimal,” and “maximal”) of stock to protect against shortages due to mis-allocation. These heuristics yield upper bounds on the optimal cost. We benchmark their performance by determining the optimality gap between the upper bounds they yield and the lower bound obtained from the relaxation.

Numerical experiments yield structural results concerning the state-dependent $(s(\hat{V}), S(\hat{V}))$ policies, some of which we prove for a special case. These experiments quantify the performance of the
heuristic – over an experiment with 540 instances, we find the average optimality gap obtained from the best heuristic $PL(\sigma)$ to be 2.08%; and identify the cost benefit of advance demand information and delivery flexibility – on average a 14.06% cost reduction was achieved in our experiments by introducing flexible delivery to an ADI system. An interesting finding is that increasing the demand leadtime by one period has a higher benefit than shortening the supply leadtime by one period. This is in contrast to previous research (Hariharan and Zipkin 1995) showing that the two are equivalent for systems with ADI but no delivery flexibility. We show that delivery flexibility and ADI are complements: The benefit of delivery flexibility is higher when there are higher degrees of advance demand availability.

The remainder of this paper is organized as follows. Section 2 positions our work in the context of the advance demand information literature. In Sections 3 and 4, we develop and analyze models with homogeneous and heterogeneous customers, respectively. Each section includes numerical analysis followed by structural and managerial insights obtained from them. Concluding remarks are presented in Section 5. All the proofs can be found in the e-companion to this paper, unless otherwise noted.

2. Literature Review

Our model directly contributes to the stream of research that analyzes uncapacitated inventory systems (where the supply leadtime is exogenous) with advance demand information and exact delivery. In addition to Hariharan and Zipkin (1995)’s continuous-review model discussed above, Gallego and Özer (2001) study a periodic-review model with heterogeneous advance demand information. They show that it is optimal to adopt a modified $(s, S)$ policy, where replenishments are made to raise the modified inventory position (=inventory position minus advance demands, hereafter MIP) to $S$ whenever MIP reaches or drops below $s$. Gallego and Özer (2003) and Özer (2003) extend this analysis to multi-echelon models, and distribution systems, respectively. Other related models that all demonstrate the benefits of ADI are Bourland et al. (1996) in a two-stage supply system, Güllü (1997) in a two-echelon, single-depot, multiple-retailer problem, Decroix and
Mookerjee (1997) in a costly information acquisition setting, and van Donselaar et al. (2001) in a project-based (pure make-to-order) environment. Lu et al. (2003) study an assemble-to-order system with stochastic leadtime and advance demand information. Our analysis establishes the structure of the optimal policy under flexible delivery with a deterministic supply leadtime, and quantifies both the magnitudes of and the interaction between the values of ADI and delivery flexibility.

An article that allows for a flexible time-window fulfillment scheme is Wang et al. (2005) that studies inventory management with a service level constraint. Assuming the inventory policy is of the $(s,S)$ type, the authors develop algorithms for searching for the optimal $(s,S)$ levels and demonstrate the trade-off between inventory cost and demand leadtime. Our model proves the optimality of state-dependent $(s,S)$ policies with respect to the modified inventory position with homogeneous customers.

A related stream of literature considers pricing and strategic interactions with ADI. Chen (2001) studies retailer market segmentation strategies with advance demand information. By offering different prices and delivery schedules, the company is able to segment customers according to different demand leadtimes. Thonemann (2002) and Zhu and Thonemann (2004) analyze the benefits of obtaining different levels of future demand information from multiple customers. Ho and Zheng (2004) provide interesting examples of flexible delivery in practice, and discuss the role of delivery time commitment and customer expectations in market competition. Tang et al. (2004) model an advance booking program for perishable seasonal products and present the optimal discounting policy to induce customers to pre-commit. McCardle et al. (2004) extend the model to a duopoly environment and identify conditions such that both retailers implementing advance booking program is the unique equilibrium. By quantifying the value of flexible delivery, our model can provide the basis for market segmentation and contract negotiation with flexible delivery.

The value of advance demand information has also been analyzed in capacitated production-inventory systems (modeled as queues, where the supply leadtime is endogenous). Buzacott and
Shanthikumar (1994) analyze a single-stage make-to-stock queue with advance demand information, and investigate the relationship between safety stock and safety leadtime. Karaesmen et al. (2002) present a discrete-time version of Buzacott and Shanthikumar (1994). They show that generalized base-stock policies are optimal and conjecture the optimality of order base-stock policies for leadtimes below a threshold; these policies are further characterized and evaluated in Karaesmen et al. (2003) and Karaesmen et al. (2004). Wijngaard and Karaesmen (2005) prove the conjecture for an M/D/1 queue. Güllü (1996), Toktay and Wein (2001), and Hu et al. (2003) use the Martingale Model of Forecast Evolution (developed in Heath and Jackson 1994, Graves et al. 1998), and Özer and Wei (2004) use additive forecast updates to model advance demand information in capacitated discrete-time production-inventory systems. They characterize or provide approximations for the optimal order base-stock level, and investigate the value of ADI.

In this stream of literature, Karaesmen et al. (2004) and Jemai (2003) are particularly relevant as they allow for early delivery, and delivery within a given time window, respectively, assuming homogeneous demand leadtimes. Based on the homogeneity of the customers, Karaesmen et al. (2004) consider a base-stock policy where arriving orders trigger immediate production releases and all outstanding orders are satisfied as soon as possible in a first-come-first-served manner. They show that the model with advance demand information and delivery flexibility is then equivalent to one with no advance demand information and a modified backorder cost. Jemai (2003) generalizes the analysis to delivery within a time window, of which the analysis in Karaesmen et al. (2004) is a special case. He shows that in decentralized production-inventory systems operating under base-stock policies, a time window contract can reduce the inefficiencies and even coordinate the system. In this paper, we also exploit the first-come-first-served characteristic of homogeneous leadtimes. This characteristic is key in showing the optimality of the modified state-dependent base-stock policy. With heterogeneous leadtimes, demand cross-over can occur, in which case we develop and evaluate approximate order and fulfilment policies.

Finally, the concept of our protection level heuristics is closely related to the inventory rationing literature (e.g. Veinott 1965, Topkis 1968, Ha 1997, de Véricourt et al. 2002). These models are
also concerned with the optimal ordering policy and how to allocate on-hand inventory to different demand classes. The optimal allocation policy normally consists of a protection level for each segment. In these models, customers differ in their sensitivity to stock-outs, represented by different shortage costs or fill rate requirements, while in our model, customers differ in their willingness to wait, represented by the demand leadtimes.

3. Analysis with a Homogeneous Customer Base

We consider a single-item, finite-horizon, periodic-review inventory system. The inventory manager makes an ordering decision at the beginning of each period to minimize discounted expected inventory holding and backorder costs over a finite planning horizon of \( N \) periods. The sequence of events in any period is as follows: inventory review, placement of new order, receipt of replenishing delivery, demand arrival, and fulfillment of demand. All quantities (eg. demand, inventory level/position, replenishment order) are assumed to be integers.

In this section, we analyze the case with a homogeneous customer base where the demand leadtimes of all customers are identical and denoted by \( T \). Demand arriving in period \( i \) is denoted by the scalar \( d_i \). This demand is due by period \( i+T \), since fulfilling it in any period within the time-window \([i, i+T]\) is considered a successful fulfillment. Partial fulfillment of an order is allowed. All unsatisfied overdue demands are fully backlogged, and a backorder penalty is applied per period. Demands in different periods are independent.

The inventory manager determines the order quantity \( z_i \) in period \( i \). The supply leadtime \( L \) is assumed to be a known nonnegative constant, which means that a replenishing order placed at the beginning of period \( i \) will arrive at the beginning of period \( i+L \). Outstanding supply arriving in period \( j \) is denoted by \( w_j \).

At the beginning of period \( i \), the system state is given by \((x_i, W_i, V_i)\). Here, the scalar \( x_i \) is the inventory level (on-hand inventory minus backorders); the vector \( W_i = (w_i, w_{i+1}, \ldots, w_{i+L-1}) \) is the supply pipeline, and vector \( V_i = (v_i^i, v_{i+1}^i, \ldots, v_{i+T-1}^i) \) is the advance demand profile, where \( v_{i,j}^i, j \geq i \) is the unsatisfied advance demand at the beginning of period \( i \) that is due by period \( j \).
We next derive the state evolution equations. The evolution of the vector $W$ is simple since there is no possible control on the supply stream: the whole pipeline just moves one position forward, and the new order is inserted in last position, i.e.,

$$W_{i+1} = (w_{i+1}, w_{i+2}, \ldots, w_{i+L-1}, z_i).$$  \hspace{1cm} (1)

However, delivery flexibility significantly modifies the dynamics of $x_i$ and $V_i$, which have a linear structure in inventory models without delivery flexibility. In basic inventory models without advance demand information, $x_{i+1}$, the inventory level at the beginning of period $i + 1$, is equal to $x_i + w_i - d_i$, where $w_i$ and $d_i$ are the replenishment quantity to be received in period $i$ and the demand arriving in period $i$, respectively. Similarly, in inventory models with advance demand information but no flexibility (see Figure 1 for a schematic representation), the evolution equation is $x_{i+1} = x_i + w_i - v^i_i$, which preserves linearity.

![Inventory Model with Advance Demand Information and a Homogeneous Customer Base](image)

In contrast, linearity is lost when flexible delivery is possible, because the inventory manager now has the freedom to satisfy future demands earlier than their due dates. In fact, if stock remains after fulfilling the current period’s demand ($x_i + w_i - v^i_i > 0$), it is optimal to ship as many of the existing orders as possible to minimize the inventory holding cost. Since the total existing orders are given by $\sum_{j=i}^{i+T-1} v^j_i + d_i$, we obtain an end-of-period inventory of $(x_i + w_i - \sum_{j=i}^{i+T-1} v^j_i - d_i)^+$. If $x_i + w_i - v^i_i < 0$, a stock-out occurs and the unfilled demand $(x_i + w_i - v^i_i)^-$ is backlogged. Here, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. Combining the two, we have

$$x_{i+1} = (x_i + w_i - \sum_{j=i}^{i+T-1} v^j_i - d_i)^+ - (x_i + w_i - v^i_i)^-. \hspace{1cm} (2)$$
Clearly, as orders arrive into the system one by one and form a serial pipeline, it is optimal to fill future orders prioritized by earliest due date (or equivalently in a first-come-first-served manner) to minimize the expected discounted backorder costs. This observation can be used to write the evolution equations for demand profile $V$:

\begin{align}
  v_{k+1}^i &= \min \left\{ (x_i + w_i - \sum_{j=i}^{k} v_j^i)^-, v_k^i \right\}, \quad k = i + 1, \ldots, i + T - 1; \\
  v_{i+T}^i &= \min \left\{ (x_i + w_i - \sum_{j=i}^{i+T-1} v_j^i - d_i)^-, d_i \right\}.
\end{align}

To interpret (3) and (4), note that three outcomes are possible. If the on-hand inventory $(x_i + w_i)$ is sufficient to cover all advance demands up to and including period $k (\sum_{j=i}^{k} v_j^i)$, then $v_{k+1}^i = 0$. If it is sufficient to cover all demands up to and including period $k - 1$ and only part of the period $k$ advance demand, then $v_{k+1}^i = \sum_{j=i}^{k} v_j^i - x_i - w_i$. Finally, if the inventory on hand can only cover at most the advance demand up to but not including period $k$, the advance demand for period $k$ is unchanged and $v_{k+1}^i = v_k^i$. Figure 2 plots $x_{i+1}$ (solid line) and $v_{k+1}^i, k = i + 1, \ldots, i + T$ (dashed lines).

Note that although the evolution of $x$ is not linear, $x - \sum v$ evolves linearly: $x_{i+1} - \sum_{j=i}^{i+T-1} v_j^i = x_i - \sum_{j=i}^{i+T-1} v_j^i + w_i - d_i$. The observation is critical when we re-define system states and collapse the dimension in the following subsections.

Applying the standard DP formulation, we can now write the optimal cost-to-go function in period $i$ as

\[ C_i(x_i, W_i, V_i) = \min_{z_i \geq 0} \left\{ c(z_i) + E_{d_i} \left[ L(x_{i+1}) + \alpha C_{i+1}(x_{i+1}, W_{i+1}, V_{i+1}) \right] \right\}. \tag{5} \]

Here, periods are indexed in increasing order. $\alpha \in [0, 1]$ is the discount factor. $c(z)$ is the ordering cost when ordering $z$ units. Since variable cost does not change the nature of the problem (Veinott 1966), we assume that there is only a fixed order cost and no variable cost, i.e.,

\[ c(z) = K \cdot 1_{\{z > 0\}}, \]

where $1_{\{A\}}$ is equal to 1 if $A$ is true, and zero otherwise. $L(x)$ is the single-period holding and backorder cost (incurred at the end of each period):

\[ L(x) = h \cdot x^+ + p \cdot x^- \].
Figure 2 State Evolution

Note. This graph demonstrates the state evolution equations as a function of $x_i + w_i$. The distance between tick marks is $v_i, v_i + 1, \ldots, v_i + T - 1, d_i$. When $x_i + w_i$ is less than $v_i$, on-hand inventory is insufficient to cover the current demand and $x_i + w_i - v_i$ units are backlogged, with advance demands $v_i + 1, \ldots, v_i + T - 1$ unchanged. At the other extreme, when $x_i + w_i$ is large enough to cover all observed demands, $x_i + w_i - \sum_{j=i}^{i+T-1} v_j - d_i$ units remain on-hand and $v_{i+1}^i, \ldots, v_{i+T}^i$ are zero. For intermediate levels of on-hand inventory, $x_i + w_i$ can cover $v_i$ and part of the other advance demands, so no inventory remains, and advance demands are fulfilled in a FCFS manner.

Note that the timing of payment may make a big difference on unit holding cost $h$. $h$ represents both physical and financial holding costs. Physical holding cost is incurred until the unit is delivered, while the financial part is incurred until the unit is paid for. Here we assume that customers pay at the time of delivery, so that the same unit holding cost (both physical and financial) will be incurred before and after customers place their orders. All cost parameters ($K, h, p$) are independent of time.

When the supply leadtime is positive, the inventory manager’s ordering decision $z_i$ has no effect on the system in periods $i, i + 1, \ldots, i + L - 1$. We adopt the standard technique of shifting the system by $L$ periods and studying the inventory level at the end of period $i + L$ (i.e., $x_{i+L+1}$). Because of the difference between the cases $T \leq L + 1$ and $T > L + 1$, we analyze their dynamics and optimal policies separately in the next two subsections.

3.1. Case 1: $T \leq L + 1$

When $T \leq L + 1$, advance demands filled by their due date are only satisfied from previously placed orders. Given the system state $(x_i, W_i, V_i)$, we can recursively derive $x_j$ and $V_j$ for all $j \in (i, T]$. Following the last remark above, we are interested in $x_{i+L+1}$. After some algebraic manipulation, we obtain
\[ x_{i+L+1} = \left( y_i - \sum_{j=i}^{i+L} d_j \right)^+ - \left( y_i - \sum_{j=i}^{i+L-T} d_j \right)^- , \]  

where \( y_i \equiv u_i + z_i \) and

\[ u_i \equiv x_i + \sum_{j=i}^{i+L-1} w_j - \sum_{j=i}^{i+L-1} v_i. \]  

Here \( u_i \) is simply the inventory position less all the advance demands, and is called \textit{Modified Inventory Position} (MIP) in period \( i \) (before order \( z_i \) is placed). \( y_i = u_i + z_i \) is then the MIP after ordering. Note that this definition differs from Gallego and Özer (2001), who define MIP as inventory position less the observed advance demands within the protection period \([t, t + L]\), i.e.,

\[ \text{MIP}_i \equiv x_i + \sum_{j=i}^{i+L-1} w_j - \sum_{j=i}^{i+L-1} v_i. \]

**Proposition 1.** For \( T \leq L + 1 \), a modified \((s, S)\) policy is optimal, where

\[ S_i = \max\{y : G_i(y) \leq G_i(x), \ \forall x\}; \]

\[ s_i = \max\{y < S_i : G_i(y) > K + G_i(S_i)\}, \]

with

\[ G_i(y_i) \equiv \alpha^L E_{d_t, \ldots, d_{i+L}} \left( y_i - \sum_{j=i}^{i+L} d_j \right)^+ - \left( y_i - \sum_{j=i}^{i+L-T} d_j \right)^- \]  

\[ + \alpha E_{d_t} f_{i+1}(y_i - d_i), \]  

\[ f_i(u_i) \equiv \min_{y_i \geq u_i} \{ K \cdot 1_{\{y_i > u_i\}} + G_i(y_i) \}. \]

Here, the system state in period \( i \) collapses to the scalar \( u_i \) with linear evolution \( u_{i+1} = u_i + z_i - d_i \) (recall the observation that \( x - \sum v \) evolves linearly). Therefore, the replenishment decision is made based on the modified inventory position. An order of \( S_i - u_i \) units should be placed whenever \( u_i \leq s_i \).

### 3.2. Case 2: \( T > L + 1 \)

In this case, it can be similarly derived that

\[ x_{i+L+1} = \left( y_i - \sum_{j=i}^{i+L} d_j \right)^+ - \left( y_i + \sum_{j=i+L+1}^{i+T} v_i \right)^-. \]
Now \( x_{i+L+1} \) can no longer be expressed by the modified inventory position and the demands only. An additional \((T-L-1)\)-dimensional vector (e.g., \( \hat{v}_i^{i+L+1}, \ldots, \hat{v}_i^{i+T-1} \)) in period \( i \) needs to be recorded. Define

\[
\hat{V}_i = (\hat{v}_i^{i+L+1}, \ldots, \hat{v}_i^{i+T-1})
\]

and let \( \hat{v}_i^j = v_i^j \) for \( j = L + 2, \ldots, T \). \( \hat{V}_1 \) is then the vector of advance demands whose due dates exceed the supply leadtime in period 1.

**Proposition 2.** For \( T > L + 1 \), a state-dependent \((s(\hat{V}), S(\hat{V}))\) policy is optimal, where

\[
S_i(\hat{V}_i) = \max\{y : G_i(y, \hat{V}_i) \leq G_i(x, \hat{V}_i), \forall x\};
\]

\[
s_i(\hat{V}_i) = \max\{y < S_i(\hat{V}_i) : G_i(y, \hat{V}_i) > K + G_i(S_i(\hat{V}_i), \hat{V}_i)\},
\]

with

\[
G_i(y_i, \hat{V}_i) = \alpha^L \sum_{d_{i+L}}^\infty E_{d_i} \cdots d_{i+L} \left( (y_i - \sum_{j=i}^{i+L} d_j)^+ - (y_i + \sum_{j=i+L+1}^{i+T-1} \hat{v}_j^i)^- \right) + \alpha E_{d_i} f_{i+1}(y_i - d_i, \hat{V}_i+1),
\]

\[
f_i(u_i, \hat{V}_i) = \min_{y_i \geq u_i} \{K \cdot 1\{y_i > u_i\} + G_i(y_i, \hat{V}_i)\}.
\]

The redefined system state variables evolve linearly: \( u_{i+1} = u_i + z_i - d_i \) and \( \hat{V}_{i+1} = (\hat{v}_{i+L+2}^i, \ldots, \hat{v}_{i+T-1}^i, d_i) \). Now the re-order point \( s_i \) and order-up-to level \( S_i \) are functions of the state vector \( \hat{V}_i \). For any given \( \hat{V}_i \), there exist two critical numbers \( s_i(\hat{V}_i) \) and \( S_i(\hat{V}_i) \) such that the modified inventory position should be raised up to \( S_i(\hat{V}_i) \) once it falls to \( s_i(\hat{V}_i) \) or below.

**3.3. Structural Results and Managerial Insights**

In this section, we implement the dynamic program for a number of experiments and point out some structural properties of the optimal policy. We then illustrate the cost benefit of advance demand information and delivery flexibility. Our main managerial insight is that the cost benefit of extending the demand leadtime is much larger than that of shrinking the supply leadtime by the same amount. This is in contrast to existing literature on advance demand information in uncapacitated systems that shows that these two are equivalent without fulfillment flexibility.
A few words on the implementation are in order. In Case 1 \((T \leq L + 1)\), the state is one-dimensional, so the DP can be solved easily. Since the model is the same as Scarf’s where the inventory position is replaced with the modified inventory position, existing algorithms to search for the optimal \((s, S)\) parameters are also readily applicable (for references, see Veinott and Wagner 1965, Zheng and Federgruen 1991). In the second case, the state has dimension \(1 + (T - L - 1)\). As solving such a high-dimensional DP is computationally prohibitive, we limit our numerical analysis to the two-dimensional case, which is the simplest non-trivial case. Specifically, we consider combinations of \((L, T)\) pairs, where \(L = 0, 1, 2, 3, 4\) and \(T = 0, 1, 2\). Among the 15 combinations, \(L = 0, T = 2\) is the only one that is two-dimensional. In this case, the vector \(\tilde{V}\) in the state \((u, \tilde{V})\) reduces to a scalar, denoted by \(\hat{v}\) below. Recall that \(u\) is the modified inventory position before ordering, and \(y\) is the modified inventory position after ordering. The optimal policy has some interesting structural properties.

**Proposition 3.** When \(T - L = 2\), the system state reduces to \((u, \hat{v})\). The following properties hold for \(i = 1, \ldots, N\):

1. The order-up-to level \(S_i(\hat{v}_i)\) is independent of \(\hat{v}_i\);
2. The re-order point \(s_i(\hat{v}_i)\) is decreasing in \(\hat{v}_i\).

The properties can be observed in the \((s_1(\hat{v}), S_1(\hat{v}))\) policy in period 1 for the case \(L = 0, T = 2\) (Figure 3), where demand has a Poisson distribution with mean \(\lambda = 6\), planning horizon \(N = 30\), discount factor \(\alpha = 1\), ordering cost \(K = 100\), holding cost \(h = 1\), and shortage cost \(p = 9\). In the figure, we can see that a replenishment is made once the MIP is below some threshold \(s_1(\hat{v})\), and the MIP is raised up to level \(S_1(\hat{v})\).

Figure 3 confirms that the order-up-to level \(S(\hat{v})\) is independent of \(\hat{v}\). The intuition is the following: By definition, the modified inventory position is equal to the inventory position minus all the known advance demands. Thus, no matter what the advance demand \(\hat{v}\) is, by raising the modified inventory position \(x_i - \hat{v}\) up to \(S\), we are able to first satisfy all the backorders and then clear all the known advance demands \(\hat{v}\), and finally have \(S\) units remaining on-hand. No
matter what \( \hat{v} \) is, the remaining on-hand inventory level \( S \), which captures the cost trade-off in the following periods, will not change. This property does not hold in Gallego and Özer (2001), where flexible delivery is not allowed. They do observe a similar pattern when \( \hat{v} \) is small, but when \( \hat{v} \) is large, since early fulfillment of \( \hat{v} \) is not possible, it does not pay to order and hold units to cover a large \( \hat{v} \): Any order one places above and beyond what can immediately be shipped to satisfy existing demand incurs inventory holding cost.

Figure 3 also shows that \( s(\hat{v}) \) is decreasing in \( \hat{v} \). In other words, as the advance demand level increases, the reorder point decreases, which may appear counterintuitive at first. To understand why, note that for a given MIP, the expected inventory holding cost is independent of \( \hat{v} \), while the expected penalty cost decreases in \( \hat{v} \), according to Equation (12). Thus a larger \( \hat{v} \) allows the inventory manager to postpone her order by choosing a lower reorder point \( s(\hat{v}) \). Gallego and Özer (2001) also observe that \( s(\hat{v}) \) is decreasing in \( \hat{v} \); this structure is not driven by delivery flexibility.

In Figure 4, we plot \( C^*_{1}(L,T) \) (in solid lines), the optimal cost as a function of \( L \) and \( T \) (where the initial state is \( (x_1, W_1, V_1) = (0, 0, 0) \)), together with the optimal cost of Gallego and Özer (2001)’s ADI model (in dashed lines). It suggests that ADI and flexibility are complements: One gains more from delivery flexibility when \( T \) becomes larger. Another interesting finding is that in our model, the cost reduction from extending the demand leadtime \( T \) to \( T + 1 \) is much larger than
that from shrinking the supply leadtime $L$ to $L - 1$. In contrast, Hariharan and Zipkin (1995) show that in their continuous-review model, what really matters is the effective leadtime $L - T$, i.e., $C^*(L, T + 1) = C^*(L - 1, T)$. In other words, the cost savings should be equal from increasing the demand leadtime or decreasing the supply leadtime by an equal amount, if there is no delivery flexibility. The symmetry is also observed in Gallego and Özer (2001)’s periodic-review ADI model (see the surface plotted in dashed lines in Figure 4). Intuitively, the only difference here is that we have delivery flexibility in our model, and the gain from such flexibility is always nonnegative. Therefore introducing delivery flexibility breaks the symmetry and favors the direction of extending the demand leadtime. This is a useful managerial insight, which says that all else being equal, effort should first be concentrated on increasing the demand leadtime. Our analysis can be particularly useful in the strategic interactions with upstream suppliers and downstream customers by providing quantitative estimates of the benefits of shortening the supply leadtime and extending the demand leadtime, which are critical when negotiating supply contracts with suppliers and when pricing delivery options for customers.

4. Analysis for a Non-Homogeneous Customer Base

We generalize our previous analysis by allowing customers to be heterogeneous in terms of demand leadtime. Specifically, there are $T + 1$ segments, with demand leadtimes ranging from 0 to $T$. In any period $i$, a demand vector $D_i = (d_i^0, \ldots, d_i^{i+T})$ is observed, where $d_i^j$ is the demand arriving in period $i$ and to be fulfilled by period $j$. We assume $D_i$’s are independent of each other. Note that the homogeneous customer case analyzed in the previous section is equivalent to $D_i = (0, \ldots, 0, d_i^{i+T})$.

The supply leadtime $L$ is again a given nonnegative constant, and the dynamic programming formulation is the same as (5), with $(x_i, W_i, V_i)$ as the system state. Now $v_i^j$ is the cumulative unsatisfied demand at the beginning of period $i$ that needs to be fulfilled by period $j$. The evolution of the supply pipeline remains the same as (1), since the supply part is unchanged. However, the evolution equations of $x_i$ and $V_i$, namely (2), (3), and (4), do not apply anymore. Remember that in writing these equations, we invoked the optimality of satisfying orders on a FCFS basis.
for homogeneous customers, which is equivalent to serving them in earliest due date order. This property no longer holds with heterogeneous customers as demand cross-over can take place: A demand arriving later can have an earlier due date than some existing orders. As a result, in addition to the ordering decision, the inventory manager now faces an allocation decision: If there is surplus inventory on hand, should she use it to satisfy observed orders that are due later in the future and reduce inventory cost, or carry the inventory over for future orders that may have earlier due dates? The answer is not straightforward. It may depend on the inventory level, cost parameters, demand distribution, etc. The key issue here is how to balance the trade-off between the holding cost that can be saved in the current period and the potential shortage costs that could be incurred in the following periods. Figure 5 provides a visual illustration of demand cross-over.

The joint optimization of ordering and allocation decisions with demand cross-over is a difficult problem, whose optimal policy could be quite complicated. In the following, we develop heuristics that are easy to implement and perform well. Note that when $T = 1$, demand $D_i$ is a two-dimensional vector $(d^i, d^{i+1})$. In this case, there is no cross-over and the problem can be solved as before. In the remainder of this section, we focus on the case $T > L + 1$, since the other case is essentially a special case where the state space reduces to one dimension.

4.1. Description of Heuristics

We first develop an approximation (AP) by introducing the allocation assumption, which is widely applied in the multi-echelon distribution system literature: We assume that units that have been
delivered to fulfill advance demands can be taken back and resent to other customers with urgent demand without incurring any costs or penalties. Mathematically, this is equivalent to allowing the delivery of negative units against advance demands (where the quantity of negative units is bounded by the quantity of positive units shipped earlier). Then the inventory manager has no reason to care about the future, since even if urgent demands arrive, she can always take the “mis-allocated” units back. Therefore, it is optimal for her to use a myopic allocation policy that uses all the on-hand inventory to satisfy the observed demands according to the earliest-due-date rule. As we demonstrate in 4.2, the allocation assumption allows us to solve for the ordering policy with previously developed techniques.

Since this approximation is a relaxation of the original problem, the cost obtained constitutes a lower bound on its optimal cost. However, the approximation is not implementable because allocated units can hardly be taken back in practice. For this reason, we propose three implementable heuristics that refine the allocation policy by introducing “protection levels” to balance the holding cost in the current period and potential shortage costs in the future. In particular, we assign protection stock $\sigma_i$ between each adjacent pair of upcoming demands $d_{i+1}^t, d_{i+2}^t, \ldots, d_{i+T}^t$. Any on-hand inventory above this protection stock level can be used to satisfy advance demands in a first-come-first serve manner. After demand is fulfilled, any remaining inventory is carried over to the next period and can be used to fulfill future urgent demands. This approach reduces shortage costs in the future, but increases the holding cost in the current period.

We develop three protection level heuristics, PL($\Sigma$), PL(0) and PL($\sigma$), as explained in detail in 4.3. PL($\Sigma$) uses a protection level that is large enough to cover the whole support of urgent demand. Because this avoids demand cross-over, we can solve a dynamic program to obtain the optimal ordering policy. PL(0) uses no protection stock and PL($\sigma$) uses an intermediate protection stock level that balances shortage and holding costs. Because of demand cross-over, we cannot solve for the optimal ordering policy for these two levels of protection stock. Instead, we use the ordering policy obtained in AP for these heuristics and evaluate their performance using simulation. Table 1 summaries all four models.
Table 1 Approximation and Protection Level Heuristics

<table>
<thead>
<tr>
<th>Model</th>
<th>Description</th>
<th>Allocation Policy</th>
<th>Ordering Policy</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>AP</td>
<td>Approximation based on allocation assumption</td>
<td>FCFS with reallocation allowed</td>
<td>$(s(\hat{V}), S(\hat{V}))$ by solving DP</td>
<td>from DP</td>
</tr>
<tr>
<td>PL(0)</td>
<td>Protection level heuristic with zero protection</td>
<td>reserve no protection stocks (equivalent to FCFS without reallocation)</td>
<td>adopt the policy of AP</td>
<td>via simulation</td>
</tr>
<tr>
<td>PL(\sigma)</td>
<td>Protection level heuristic with balanced protection</td>
<td>reserve “optimal” units of protection stocks and then fulfill the demands with the surplus</td>
<td>adopt the policy of AP</td>
<td>via simulation</td>
</tr>
<tr>
<td>PL(\Sigma)</td>
<td>Protection level heuristic with maximal protection</td>
<td>reserve “maximum” units protection stocks and then fulfill the demands with the surplus</td>
<td>$(s(\hat{V}), S(\hat{V}))$ by solving DP</td>
<td>from DP</td>
</tr>
</tbody>
</table>

The protection-level heuristics provide upper bounds on the cost of the original problem. By comparing them with the lower-bound obtained from AP, we are able to benchmark the optimality gaps. Numerical analysis is presented in 4.4, and is the basis for structural and managerial insights.

4.2. Approximation Based on the Allocation Assumption

As explained earlier, we assume that units that have been delivered to fulfill advance demands can be taken back and re-sent to other customers without incurring any costs or penalties. Mathematically, this is equivalent to allowing the delivery of negative units against advance demands.

Our assumption parallels the allocation assumption made in the analysis of multi-echelon distribution models. It is well known that the decomposition method by Clark and Scarf (1960) can be applied to serial and assembly systems, but not to distribution systems, due to the additional decision on how to allocate inventory to multiple downstream retailers optimally. Eppen and Schrage (1981) derive a closed-form optimal policy for a distribution system by making what they call the “allocation assumption.” The idea is essentially relaxing the nonnegativity constraints on allocation variables, i.e., negative delivery is allowed. Then, the allocation problem is straightforward — myopic allocation to minimize the expected cost in the current period without considering the future. Federgruen and Zipkin (1984) also make the same assumption to solve allocation problems in a similar context. Özer (2003) studies a distribution system with ADI, and once again relaxes the nonnegativity constraint. To the best of our knowledge, the allocation assumption is still the key to solve such problems, and it is believed that in general it will not hurt system performance significantly (Doğru et al. 2005).

Given the allocation assumption, the inventory level at the beginning of period $i + L + 1$ will be
\[ x_{i+L+1} = \left( y_i - \sum_{l=i}^{i+L} \sum_{j=l}^{i+T} d_l^j \right)^+ - \left( y_i + \sum_{j=i}^{i+L} \hat{v}_j^i - \sum_{l=i}^{i+L} \sum_{j=l}^{i+T} d_l^j \right)^- \]  \tag{14}

where \( u_i \) and \( \hat{V}_i \) are defined in (7) and (11), and \( \hat{v}_j^i = v_j^i \) for \( j = L + 2, \ldots, T \).

The DP can be formulated similarly as in \$3.2\), where the only difference is that the state \((u_i, \hat{V}_i)\) evolves as follows:

\[
u_{i+1} = y_i - \sum_{j=i}^{i+T} d_l^j, \tag{15}\]

\[
\hat{v}_{i+1}^j = \hat{v}_j^i + d_l^j, \quad j = i + L + 2, \ldots, i + T, \tag{16}\]

\[
\hat{v}_{i+1}^{i+T} = d_{i+T}. \tag{17}\]

The single-period loss function \( EL(x_{i+L+1}(y_i, \hat{V}_i)) \) is convex in \( y_i \) for any given \( \hat{V}_i \), so the optimality of the state-dependent \((s(\hat{V}), S(\hat{V}))\) policy is preserved. The technical details can be found in the e-companion EC.4.

### 4.3. Protection Level Heuristics

As discussed above, we propose Protection Level heuristics where protection stocks are kept against urgent demand. In particular, we assign protection stocks between each adjacent pair of upcoming demands \( d_{i+1}^i, d_{i+2}^i, \ldots, d_{i+T}^i \). To understand how this works, consider the simplest case where \( T = 2 \), where a single protection stock is sufficient. The allocation policy works as follows: Given inventory is available, first satisfy the demands due in the current period \((v_i^i + d_i^i)\) and the next period \((v_i^{i+1} + d_i^{i+1})\) (neither of these demands will be crossed over by future demands); then, if anything remains, reserve some units, \( \sigma_i \), as safety stock in period \( i \) to protect from being unable to satisfy \( d_{i+1}^{i+1} \) in the next period; finally use the surplus, if any, to fill the remaining non-urgent advance demands \( d_{i+2}^{i+2} \).

When protection levels are used, the system states evolve in a much more complicated manner. In the following, we demonstrate the case with \( T = 2 \) and \( L = 0 \). For other cases where \( T > 2 \) and/or \( L > 0 \), the result still follows, but the notation becomes very cumbersome.

There are \( x_i \) units on-hand at the beginning of period \( i \), and \( z_i \) units are ordered and arrive immediately (since \( L = 0 \)), bringing the total available inventory to \( x_i + z_i \). The observed advance
demand profile is \((v_i^i, v_i^{i+1})\), and the demand vector arriving in period \(i\) is \(D_i = (d_i^i, d_i^{i+1}, d_i^{i+2})\). As the level of available inventory varies, there could be five different situations:

1. \(x_i + z_i - v_i^i - d_i^i \leq 0\). \(v_i^i + d_i^i\) is the amount due in period \(i\), and cannot be fully satisfied from inventory. The unsatisfied quantity will be backlogged, \(x_{i+1} = x_i + z_i - v_i^i - d_i^i\), and incur backordering penalty. Other components in the demand pipeline are unchanged, so \(v_{i+1}^{i+1} = v_{i+1}^{i+1} + d_{i+1}^{i+1}\) and \(v_{i+1}^{i+2} = d_{i+1}^{i+2}\).

2. \(0 \leq x_i + z_i - v_i^i - d_i^i < v_i^{i+1} + d_i^{i+1}\). Now the inventory is enough to cover the current period’s demand, while the surplus can all be used to satisfy part of the demand due in the next period. So \(x_{i+1} = 0\), and \(v_{i+1}^{i+1} = -(x_i + z_i - v_i^i - d_i^i - v_i^{i+1} - d_i^{i+1})\), the remaining demand of the next period. \(v_{i+1}^{i+2} = d_{i+1}^{i+2}\) again.

3. \(0 \leq x_i + z_i - v_i^i - d_i^i - v_i^{i+1} - d_i^{i+1} < \sigma_i\). The inventory level is high enough so that all the demand due in period \(i\) and \(i+1\) can be covered, but the surplus is less than \(\sigma_i\), the protection level. The surplus is carried to next period and \(d_i^{i+2}\) is not satisfied. So \(x_{i+1} = x_i + z_i - v_i^i - d_i^i - v_i^{i+1} - d_i^{i+1}\), \(v_{i+1}^{i+1} = 0\), and \(v_{i+1}^{i+2} = d_{i+1}^{i+2}\).

4. \(0 \leq x_i + z_i - v_i^i - d_i^i - v_i^{i+1} - d_i^{i+1} - \sigma_i < d_i^{i+2}\). The inventory level is even higher, and the surplus, after satisfying demand in \(i\) and \(i+1\), is more than \(\sigma_i\). Then only \(\sigma_i\) units are carried to period \(i+1\), while the remaining quantity, which is less than \(d_i^{i+2}\), is delivered to fulfill \(d_i^{i+2}\) partially. So \(x_{i+1} = \sigma_i\), \(v_{i+1}^{i+1} = 0\), and \(v_{i+1}^{i+2} = -(x_i + z_i - v_i^i - d_i^i - v_i^{i+1} - d_i^{i+1} - \sigma_i - d_i^{i+2})\).

5. \(x_i + z_i - v_i^i - v_i^{i+1} - \sum_{j=i}^{i+2} d_i^{j} - \sigma_i \geq 0\). The inventory level is so high that all the demand due in periods \(i\), \(i+1\) and \(i+2\) can be satisfied, and there are still more than \(\sigma_i\) units remaining for protection from being penalized in period \(i+1\). So \(x_{i+1} = x_i + z_i - v_i^i - v_i^{i+1} - \sum_{j=i}^{i+2} d_i^{j}\) and \(v_{i+1}^{i+1} = v_{i+1}^{i+2} = 0\).

To summarize, we have the following state evolution equations, where \((x_i, v_i^i, v_i^{i+1})\) is the system state; see Figure 6 for a graphical demonstration.

\[
\begin{align*}
    x_{i+1} &= \begin{cases} 
        x_i + z_i - v_i^i - d_i^i & \text{if (1);} \\
        0 & \text{if (2);} \\
        x_i + z_i - v_i^i - d_i^i - v_i^{i+1} - d_i^{i+1} & \text{if (3);} \\
        \sigma_i & \text{if (4);} \\
        x_i + z_i - v_i^i - v_i^{i+1} - \sum_{j=i}^{i+2} d_i^{j} & \text{if (5),}
    \end{cases}
\end{align*}
\]
Note. This graph demonstrates the state variables in period $i+1$ as functions of $x_i + z_i - v_i - d_i$. The horizontal axis can be segmented into five intervals (1) to (5), corresponding to the five cases discussed above. $x_{i+1}$ is the piecewise linear function plotted with a solid line, and $v_{i+1}^{i+1}$ and $v_{i+1}^{i+2}$ are the two dashed lines.

$$ (v_{i+1}^{i+1}, v_{i+1}^{i+2}) = \begin{cases} 
(v_i^{i+1} + d_i^{i+1}, d_i^{i+2}) & \text{if (1)}; \\
(-(x_i + z_i - v_i^i - d_i^i - v_i^{i+1} - d_i^{i+1}), d_i^{i+2}) & \text{if (2)}; \\
(0, d_i^{i+2}) & \text{if (3)}; \\
(0, - (x_i + z_i - v_i - d_i^i - v_i^{i+1} - d_i^{i+1} - \sigma_i - d_i^{i+2})) & \text{if (4)}; \\
(0, 0) & \text{if (5)}. 
\end{cases} $$

Ideally, we would jointly optimize the ordering decision variable $z_i$ and the allocation decision variable $\sigma_i$, or at least determine the optimal ordering policy for given protection levels. Unfortunately, we find that the previous technique to reduce the dimensionality and reformulate the DP with state $(u_i, \hat{V}_i)$ no longer applies, unless demand cross-over is fully avoided (this requires that $\sigma_i$ to cover the whole support of $d_i^{i+1}$). This suggests the following heuristic that we call PL($\Sigma$).

**Heuristic PL($\Sigma$).** In this heuristic, we take the protection level high enough to cover the whole support of the urgent demand ($d_i^{i+1}$), or if the support is infinite, large enough to make the probability that $d_i^{i+1} > \sigma_i$ arbitrarily small. As demand cross-over is avoided in this manner, it can be shown that the optimal ordering policy is still a modified state-dependent ($s(\hat{v}), S(\hat{v})$) policy, if the demand probability density/mass functions are strongly unimodal$^1$ (or equivalently, log-concave). The formulation and proof of this result can be found in the e-companion.

$^1$Most commonly used distributions (e.g., uniform, normal, Poisson, binomial) are strongly unimodal, see Dharmadhikari and Joag-dev (1988) for more details.
Heuristic PL(0). With zero protection stock, we recover the myopic allocation policy (without re-allocation). For the ordering policy, we use the one obtained in AP. Federgruen and Zipkin (1984) show that the policy obtained ((s, S) ordering policy and myopic allocation policy) in their approximation model for distribution systems is near-optimal. In contrast, myopic allocation, which minimizes inventory holding cost but omits possible future shortage cost, could potentially be far from optimal in our model. The cost of this heuristic is obtained via simulation.

Heuristic PL(\sigma). Clearly, PL(\Sigma) and PL(0) are the two extremes: the former heuristic avoids future shortage costs without considering the holding costs imposed by the protection stocks, while the latter minimizes the holding costs but omits the shortage costs. These policies may perform well under some extreme settings (such as very low holding or penalty cost), while for others, a properly chosen protection level that balances the two costs would be preferable. This motivates the PL(\sigma) allocation policy, where protection level \sigma_i is chosen to minimize a specific newsvendor-like objective function

$$ H(\sigma_i) = h \cdot \sigma_i + p \cdot \mathbb{E}[(d_{i+1}^{i+1} - \sigma_i)^+]. $$

(20)

This is based on the observation that the protection level affects cost only when the inventory level is in region (4) of Figure 6. In that case, increasing \sigma_i by one unit incurs one unit of holding cost for sure, but incurs penalty cost only if in the next period, the protection stock plus the arriving replenishment \( z_{i+1} \) is not enough to cover the urgent demand \( d_{i+1}^{i+1} \). Here \( z_{i+1} \) is difficult to estimate or predict, so we conservatively take it as zero to obtain (20). Essentially, the PL(0) heuristic minimizes the first term of \( H(\cdot) \) by setting zero protection levels, PL(\Sigma) minimizes the second term by setting protection levels large enough to cover \( d_{i+1}^{i+1} \), and PL(\sigma) strikes a balance between the two parts by setting \( \sigma_i \) to minimize \( H(\cdot) \).

Equation (20) is presented for the \( T = 2 \) case. When \( T > 2 \), more than one protection level are needed. The protection levels can be defined similarly. For example, when \( T = 3 \), we need protection stock, \( \sigma_{i+1}^{i+1} \), covering future demand \( d_{i+1}^{i+1} \) and protection stock, \( \sigma_{i+2}^{i+2} \), covering \( d_{i+1}^{i+2} \).
\( \sigma_{i+1} \) can still be defined as the minimizer of (20), while \( \sigma_{i+2} \) can be defined as the minimizer of

\[
h \cdot \sigma_{i+2} + p \cdot E[(d_{i+1}^{i+1} + d_{i+1}^{i+2} - \sigma_{i+1}^{i+1} - \sigma_{i+2}^{i+2})^+].
\]

Note that the function \( H(\cdot) \) is just one of many that could capture the trade-off between the holding cost due to keeping protection stocks and the penalty cost due to not being able to satisfy \( d_{i+1} \). \( H(\cdot) \) can be further refined, for example, by incorporating the effect of the fixed ordering cost \( K \) (hence indirectly capturing the effect of \( z_{i+1} \)), inventory level \( x_i \), or even the whole system state.

Nevertheless, (20) is simple, it provides a stationary protection level that is easy to implement, and it captures the most critical trade-off between the holding cost and the penalty cost.

As mentioned before, since the allocation policy \( PL(\sigma) \) does not rule out cross-over, it is difficult, if not impossible, to solve for the optimal ordering policy and estimate its cost analytically. Instead, we use the ordering policy obtained in the approximation model, based on the belief that the optimal ordering policies are relatively insensitive to different allocation policies adopted, and calculate the cost of \( PL(\sigma) \) via simulation. The robustness of the ordering policy to the allocation rules and the performance of the heuristics are tested in the next subsection.

### 4.4. Structural Results and Managerial Insights

To quantify the impact of delivery flexibility, and to evaluate the performance of the proposed heuristics, we start by mimicking an experiment in Gallego and Özer (2001), found in Table 3 of their paper. All the cost parameters remain the same as the previous numerical analysis, i.e., \( K = 100, h = 1, p = 9 \), and we again focus on the case where \( L = 0 \) and \( T = 2 \). The planning horizon is 12 periods. The discount factor is \( \alpha = 1 \).

Differing from the homogeneous model, now the demand is a 3-dimensional vector of Poisson random variables with mean \((\lambda_0, \lambda_1, \lambda_2)\). We follow the setting in Gallego and Özer (2001), where the total demand rate is constant, with \( \lambda_0 + \lambda_1 + \lambda_2 = 6 \), and the demand scenario \((\lambda_0, \lambda_1, \lambda_2)\) varies from \((5, 1, 0)\) to \((4, 1, 1)\), ..., to \((0, 1, 5)\) (labeled Expr 1 to 6). In this manner, different degrees of advance demand information availability can be modeled, and their benefits can be measured. One extreme case is \((6, 0, 0)\) (labeled Expr 0), which can be regarded as the traditional case where no advance demand information is available at all. The other
is \((0,0,6)\) (labeled Expr 7), which is exactly the homogeneous case we considered in the previous section.

**Properties of the Ordering Policies.** We first calculate the optimal ordering policies for the approximation (AP for short) and the \(PL(\Sigma)\) heuristic, which can be analyzed by solving the corresponding dynamic programs. Since \(PL(\Sigma)\) requires a “large enough” protection level, while Poisson demand is unbounded, we set \(\Sigma\) to be such that \(P\{d_i > \Sigma\} < 0.001\). Table 2 reports the protection levels used in \(PL(\Sigma)\), as well as \(PL(0)\) and \(PL(\sigma)\) for future reference.

Table 2 Protection Levels

<table>
<thead>
<tr>
<th>Expr. No.</th>
<th>(\lambda_0)</th>
<th>(\lambda_1)</th>
<th>(\lambda_2)</th>
<th>Protection Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

\[L = 0, T = 2, K = 100, h = 1, p = 9, N = 12\]

Table 3 shows the period-1 ordering policy for AP in all six demand scenarios (Expr 1 to 6). Optimal ordering policies for Expr 0 and 7 are also reported for ease of comparison. It is striking that when we compute the optimal policies of heuristic \(PL(\Sigma)\), we find that they are exactly the same as those in Table 3. This provides some support for our conjecture that the optimal ordering policies are insensitive to the allocation policy.

Once again, we observe that \(S(\hat{v})\) is independent of \(\hat{v}\) and \(s(\hat{v})\) is decreasing in \(\hat{v}\). We also observe that both the order-up-to level and the re-order point decrease as advance demand information becomes available earlier (as we progress from Expr 0 to 7).

**Performance of the Heuristics.** For the same parameter set as above, we now compare six models for a range of advance demand information scenarios (Expr 1-6): (1) traditional inventory model without ADI, (2) inventory model with ADI, (3) the approximation of ADI with flexible delivery (“ADI-F AP”), (4) PL(0) heuristic, (5) PL(\(\sigma\)) heuristic, and (6) PL(\(\Sigma\)) heuristic. Among
them, the costs of PL(0) and PL(σ) are obtained by averaging 100,000 runs of simulation of the 12-period problem, while others are obtained by solving the corresponding dynamic programs.

Figure 7 shows the results. First, by examining the gaps between (1) and (2) and between (2) and (3)-(6), we can easily see the cost benefit of ADI and of delivery flexibility, respectively. As more advance demand information becomes available (going from Experiment 1 to 6), the value of advance demand information and the value of flexibility both get increasingly large. Second, by
comparing heuristics (4), (5), and (6) with the lower bound (3), we can identify the optimality gaps.

As expected, the myopic allocation policy (PL(0)) performs the worst. Although PL(σ) dominates PL(Σ), both are quite close to the lower bound. Note that all three heuristics coincide with the lower bound and yield the optimal cost in Expr 1 and 6 because of the absence of cross-over in these experiments.

Next, we extensively test the performance of the three heuristics by comparing them with the lower bound obtained from AP over a range of parameter values. The costs of PL(0), PL(σ), PL(Σ), and AP are denoted by $C^*_0$, $C^*_σ$, $C^*_Σ$, and $C^*_AP$, respectively. Define the optimality gaps

$$δ_0 = \frac{C^*_0 - C^*_AP}{C^*_AP} \times 100\%,$$

$$δ_σ = \frac{C^*_σ - C^*_AP}{C^*_AP} \times 100\%,$$

$$δ_Σ = \frac{C^*_Σ - C^*_AP}{C^*_AP} \times 100\%.$$

Demand scenarios Expr 1 and 6 are eliminated since there is no cross-over and all heuristics yield the optimal solution. For the remaining demand scenarios, we run experiments with $L = 0, 1, 2, 3, 4$, $K = 50, 100, 200$, $h = 1, 3, 5$, and $p = 9, 19, 29$, for a total of 540 experiments. As summarized in Table 4, we find all three heuristics perform well with a small average optimality gap (less than 5%). PL(0) and PL(Σ) show similar performance, while PL(σ) appears to be the best, with the mean and standard deviation of $δ_σ$ equal to close to half of those of $δ_0$ and $δ_Σ$.

**Table 4** Optimal Gaps

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>st. dev.</th>
<th>min</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>$δ_0$</td>
<td>4.83%</td>
<td>5.03%</td>
<td>0.29%</td>
<td>38.39%</td>
</tr>
<tr>
<td>$δ_σ$</td>
<td>2.08%</td>
<td>2.25%</td>
<td>0.01%</td>
<td>15.97%</td>
</tr>
<tr>
<td>$δ_Σ$</td>
<td>4.13%</td>
<td>4.36%</td>
<td>0.07%</td>
<td>27.24%</td>
</tr>
</tbody>
</table>

**Figure 8** Optimal Gaps of the Heuristics (Subtotaled Average)
To understand the drivers of these gaps and of their differences, Figure 8 plots the subtotaled average of $\delta_\sigma$ (solid bar), $\delta_\Sigma$ (hatched bar), and $\delta_0$ (light bar) corresponding to different parameters. For example, the first figure plots the average of $\delta_\sigma$, $\delta_\Sigma$, and $\delta_0$ (each averaged over 135 experiments) for Expr 2 to 5; the last figure plots the average optimality gap for $p = 9, 19$ and $29$, respectively, each of which is obtained by averaging over 180 experiments. It appears that under all the settings, PL($\sigma$) outperforms the other two heuristics with a roughly 50% smaller optimality gap, while PL(0) and PL($\Sigma$) can also be good under some but not all settings.

The impact of advance demand availability is intuitive. When the urgent ($\lambda_0$) and non-urgent ($\lambda_2$) demands are of closer magnitude (with the total expected demand constant), the trade-off between satisfying non-urgent demand in advance and keeping inventory for future urgent demand is more salient, and the loss due to mis-allocation can be more significant. This explains why the optimality gaps are higher in the middle (Expr 3,4,5) and diminish at both extremes (Expr 1,2 and 6). The PL($\sigma$) heuristic balances the two costs and therefore shows the least optimality gap.

When $h$ becomes larger, all three heuristics deteriorate. PL($\Sigma$) deteriorates most quickly, since the heuristic offers full protection against shortage by holding a high level of inventory. The optimality gap of PL(0) increases slowly, since it minimizes holding cost with no consideration of possible shortage. As $\sigma_i$ is chosen to balance the holding and shortage costs, $\delta_\sigma$ increases at a moderate rate, and the gap is mainly due to the over-protection that comes from not considering future replenishments when choosing $\sigma_i$. When $p$ increases, $\delta_\Sigma$ decreases, $\delta_0$ increases, and $\delta_\sigma$ appears to be insensitive in $p$. These can be similarly explained by the different focuses of the heuristics.

As discussed earlier, PL($\Sigma$) and PL(0) may also perform well under some extreme settings. When $h$ is low ($h = 1$), the cost of holding protection stock conservatively is low, so $\delta_\Sigma$ is quite close to $\delta_\sigma$ and both are far smaller than $\delta_0$. On the other hand, when $p$ is low ($p = 9$), the cost of being myopic is less significant, so PL(0) exhibits a lower optimality gap than PL($\Sigma$).

While PL($\Sigma$) and PL(0) dominate one another depending on problem parameters, the PL($\sigma$) heuristic works the best in all experiments with a reasonably low average optimality gap of 2.08%. More importantly, the stationary protection level it adopts is easy to implement.
Impact of Delivery Flexibility. To provide a counterpart to Figure 4, we plot in Figure 9 the costs of ADI (in dashed lines) and ADI-F (in particular, the PL(σ) heuristic, in solid lines) under Expr 1-6, with $K = 100, h = 1, p = 9$, and supply leadtime $L$ ranging from 0 to 4. Notice here we are not comparing the cases $T = 0, 1, 2$ as we did in Figure 4 to capture degrees of advance demand information availability, instead, $T$ is constant at 2 but the degree of ADI is captured by varying the demand scenarios from Expr 1 to 6. The gap between the two surfaces, which represents the benefit due to delivery flexibility, is increasing (almost linearly) as more advance demand information becomes available (moving from Expr 1 to 6). This is consistent with the previous observation in Figure 4 that advance demand information and delivery flexibility are complements.

Figure 9 is for the base case with $K = 100, h = 1$ and $p = 9$. We next consider demand scenarios 1 through 6 with combinations of $L = 0, 1, 2, 3, 4$, $K = 50, 100, 200$, $h = 1, 3, 5$, and $p = 9, 19, 29$, for a total of 810 experiments. The costs of PL(σ), and AP are compared with the cost of ADI, denoted by $C^*_{ADI}$. Define $\Delta$, the percentage cost savings due to introducing delivery flexibility and using the PL(σ) heuristic, and its upper bound $\overline{\Delta}$ with

$$\Delta = \frac{C^*_{ADI} - C^*_\sigma}{C^*_{ADI}} \times 100\% \quad \text{and} \quad \overline{\Delta} = \frac{C^*_{ADI} - C^*_AP}{C^*_{ADI}} \times 100\%.$$ 

Summarizing all 810 experiments, $\Delta$ ($\overline{\Delta}$) ranges from 0.32% (0.37%) to 49.42% (49.43%), with mean 14.06% (15.13%) and standard deviation 10.67% (11.21%). In other words, benefits from
delivery flexibility can be substantial.

![Cost Savings of Delivery Flexibility (Subtotaled Average)](image)

Figure 10 plots the subtotaled average $\Delta$ (height of the darker bars) and $\overline{\Delta}$ (full height of the bars) for different parameters and demand scenarios. Notice that for Expr 1 and 6, $\Delta$ and $\overline{\Delta}$ are the same since there is no demand cross-over and both the approximation and the heuristic are optimal.

Figure 10 provides several insightful findings. We observe that delivery flexibility is more beneficial when advance demand availability is high. We already know that cost savings increase as more advance demand information is available, even without flexibility. If in addition, delivery flexibility is allowed, cases with higher degrees of advance demand availability post higher gains from flexibility (on top of the savings due to ADI), as seen in Figure 10. This highlights that flexibility and advance demand information are complements.

In addition, delivery flexibility is more beneficial when the holding cost $h$ is large, and is insensitive to the backorder cost $p$. The results are quite intuitive: When compared with the pure ADI model, the additional delivery flexibility mainly helps reduce the holding cost and does not have significant contributions to reducing the shortage cost.

5. **Concluding Remarks**

In this paper, we incorporate flexible delivery into inventory models with advance demand information. Flexible delivery is particularly relevant in settings when the firm sells directly to end consumers, who may be quite content to receive their orders before the quoted due date. The majority of the existing literature is motivated by industrial settings where the customer is a distributor...
or a manufacturer who derives no utility but rather incurs additional cost from early delivery. With the explosive increase in direct customer sales, analyzing inventory policies with flexible delivery has become particularly relevant. Our analysis makes several contributions in this domain, both technical and managerial.

We show that the optimal policy is a state-dependent \((s, S)\) policy when customer demand lead-times are homogeneous, and discuss how its characteristics differ from the case without flexible delivery. In particular, we find that increasing the demand leadtime is more beneficial than decreasing the supply leadtime; this is in contrast to existing results that demonstrate the equivalence of the two when there is no delivery flexibility. This is a useful managerial insight, which says that all else being equal, effort should first be concentrated on increasing the demand leadtime. The supply leadtime is primarily due to physical factors (such as production and transportation delay), and is more of an operational concern. The demand leadtime, on the other hand, is more a marketing and sales issue. Our analysis, from the perspective of the operations-marketing interface, illustrates how the two jointly determine system performance. It further quantifies the marginal benefits of shortening the supply leadtime and extending the demand leadtime, which is valuable when negotiating supply contracts with suppliers and when pricing delivery options for customers.

For heterogeneous customers, we propose a tractable approximation and implementable heuristics. In an experiment with 540 instances, we find that the most refined heuristic has an average optimality gap of 2.08%. It is already known that advance demand information is a powerful tool in reducing inventory costs. Our numerical study shows that flexible delivery can provide significant additional benefits: In our data set, a 14.06% average cost reduction is obtained from introducing flexible delivery to an inventory system with advance demand information.

Our numerical study also reveals that advance information and flexibility are complements: The more advance information is available, the more the additional value obtained from allowing flexible delivery. The implication is that it is especially beneficial to use flexible delivery in conjunction with a menu of delivery leadtimes (e.g. 2-day, 1-week and 14 day options), compared to offering all
customers a uniformly short delivery leadtime. We also find that the relative benefit of flexibility increases as the inventory holding cost increases, but is rather insensitive to penalty cost.

Before concluding, we discuss the implications of relaxing some assumptions in our model. In general, firms incur both physical and financial holding costs – the physical holding cost is incurred until the unit is delivered, while the financial part is incurred until the unit is paid for. Our model assumes that customers pay at the time of delivery so that the accumulation of the two types of cost both stop at the time of delivery. Then the same unit holding cost (both physical and financial) is incurred before and after customers place their orders, which allows for a simple formulation with holding cost parameter $h$. If customers pay either upon ordering or at the agreed upon due date regardless of the actual delivery time, then the supplier still saves the physical holding cost by delivering earlier than the due date. Therefore, our qualitative results concerning the value of flexibility, the impact of $L$ and $T$, and the sensitivity of the various heuristics to problem parameters are not affected by the timing of payment because delivery flexibility impacts at least the physical holding cost of inventory. The magnitude of the supplier’s cost, however, is clearly different under the three scenarios (with payment upon ordering being the best). We also note that introducing delivery flexibility for a given payment structure benefits the supplier the most when payments are made at the time of delivery as it creates savings on both the financial and the physical costs of holding inventory.

Second, we assume the same backorder cost for customers with different delivery leadtimes. This is a critical assumption to limit the state space. At the same time, we expect that in practice, it is most likely that ‘impatient’ customers with a shorter demand leadtime will have the higher backorder costs. If this is true, then the priority rule will not change in the allocation problem – we would still want to fulfill demand according to the earliest-due-date rule. In this case, it appears reasonable to approximate the original problem by implementing our model with the (weighted) average backorder cost as input.

Advance demand information has been studied in serial systems, single-depot multiple-retailer systems, and capacitated systems from different perspectives. Incorporating delivery flexibility in
these models is an obvious next step. We discussed how increasing the demand leadtime under flexibility is more beneficial than decreasing the supply leadtime. Since asking customers to wait longer can reduce customer satisfaction, the supplier may need share some of the cost benefit with its customers by offering different levels of price discounts to orders with larger due dates. The optimal pricing policy is an interesting direction for future research. Although we argued that delivery flexibility is primarily relevant in direct sales to end customers, it can also be valuable in industrial settings. Cohen et al. (2003) argue that between buyers and manufacturers of semiconductor manufacturing equipment, total supply chain performance could be improved if the buyers were willing to share some of the holding cost by accepting early delivery of equipment. Our model can be used to design an early delivery contract between a manufacturer and a buyer in settings where both parties incur significant holding costs from ownership of the product.

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References


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EC.1. Preliminaries

EC.1.1. K-Convexity

**Definition EC.1.** (Gallego and Özer 2001) A function $g : \mathbb{R} \rightarrow \mathbb{R}$ is $(a, b)$-convex ($a \geq 0, b \geq 0$), denoted by $g \in C(a, b)$, if

$$
g(\theta x_1 + (1 - \theta) x_2) \leq \theta [a + g(x_1)] + (1 - \theta) [b + g(x_2)], \text{ for all } x_1 \leq x_2, \theta \in [0, 1].$$

**Lemma EC.1.** (Gallego and Özer 2001) $C(a, b)$ functions have the following properties:

1. $C(a, b) \subset C(a', b')$ for all $(a, b) \leq (a', b')$.
2. If $f \in C(a, b)$ and $g \in C(a', b')$, then for positive constants $\alpha$ and $\beta$, $\alpha f + \beta g \in C(\alpha a + \beta a', \alpha b + \beta b')$.
3. If $f \in C(a, b)$ and $E[|f(x - D)|] < \infty$, then $F(x) = E[f(x - D)] \in C(a, b)$.
4. If $f(x, y) \in C(a, b)$ for a fixed vector $y$, and $E_D[|f(x - D, y)|] < \infty$, then $F(x) = E_{D,Y}[f(x - D, y)] \in C(a, b)$ where $Y$ is a vector of random variable.

The above lemma is a standard result in the literature, and is used in our proofs of optimality. For simplicity of notation, we use $(0, K)$-convex and $K$-convex interchangeably.

**EC.1.2. Strong Unimodality**

**Definition EC.2.** (An 1998) Suppose $g(x) : \mathbb{R}^k \rightarrow \mathbb{R}_+$ is a measurable function with $\mathcal{B} = \{x \in \mathbb{R}^k : g(x) > 0\}$. $g(x)$ is logconcave (logconvex) if $\log g(x)$ is concave (convex), i.e.,

$$
g(\lambda x_1 + (1 - \lambda) x_2) \geq (\leq) [g(x_1)]^\lambda [g(x_2)]^{1-\lambda},$$

for all $x_1, x_2 \in \mathcal{B}$ and all $\lambda \in [0, 1]$.

A vector of random variables, $Y$, is logconcavely (logconvexly) distributed (or equivalently, strong unimodal) if its density function $\phi(y)$ is logconcave (logconvex) on the distribution support $\Omega$.

**Lemma EC.2.** (Dharmadhikari and Joag-dev 1988, Thm 1.10) $g(x) = E_y[f(x - y)]$ is unimodal for any unimodal function $f$ if $y$ is strong unimodal (logconcavely distributed).

**Lemma EC.3.** $g(x) = E_y[f(x - y)]$ is quasiconvex for any quasiconvex function $f$ if the $y$ is logconcavely distributed.
Proof. By Lemma EC.2, \(-g(x) = -E_y[f(x-y)] = E_y[-f(x-y)]\) is unimodal since \(-f\) is unimodal. Thus \(g\) is quasiconvex. □

**EC.1.3. K-Quasiconvexity and K-Monotonicity**

**Definition EC.3.** (Gallego 1998) A function \(g\) is \((a,b)\)-quasiconvex, denoted by \(g \in QC(a,b)\), if
\[
g(x) \leq \max(g(x_1) + a, g(x_2) + b), \forall x_1 \leq x \leq x_2.
\]

\((0,K)\)-quasiconvex is also written as \(K\)-quasiconvex for short.

**Definition EC.4.** (Gallego 1998) A function \(g\) is \(K\)-increasing, denoted by \(g \in I(0,K)\), if
\[
g(x) \leq g(y) + K, \forall x \leq y.
\]

The following lemmas are stated without proofs (Gallego 1998).

**Lemma EC.4.** If \(g \in I(0,K)\) and \(h \in I(0,K')\), then \(g + h \in I(0,K + K')\).

**Lemma EC.5.** If \(g(x)\) is decreasing on \(x \leq a\) and \(g \in I(0,K)\) on \(x \geq a\), then \(g \in QC(0,K)\).

In words, the first one says that the \(K\)-increasing property is preserved through convex combination, and the second one characterizes the relation between \(K\)-monotonicity and \(K\)-quasiconvexity.

**EC.2. Homogeneous Customer Base – Case 1: \(T \leq L + 1\)**

**EC.2.1. Proof of Proposition 1**

We first derive (6) in more detail. Given system state \((x_i, W_i, V_i)\) and the evolution equations (1), (2), (3), and (4), we can derive
\[
x_{i+L+1} = \left( x_i + \sum_{j=i}^{i+L-1} w_j + z_i - \sum_{j=i}^{i+T-1} v_i^j - \sum_{j=i}^{i+L} d_j \right)^{+} - \left( x_i + \sum_{j=i}^{i+L-1} w_j + z_i - \sum_{j=i}^{i+T-1} v_i^j - \sum_{j=i}^{i+L-T} d_j \right)^{-}.
\]

Although this equation seems complicated, the logic behind it is rather straightforward: \(x_i\) is the beginning inventory in period \(i\), \(\sum_{j=i}^{i+L-1} w_j + z_i\) is the total replenishment arriving in periods \([i, i+L]\), and \(\sum_{j=i}^{i+T-1} v_i^j\) is the sum of advance demands in the pipeline in period \(i\), which are all due in or before period \(i + L\) (since \(T \leq L + 1\)). The demands arriving after period \(i\) can be separated into two parts, those due by \(i + L\) (i.e., \(d_i, \ldots, d_{i+L-T}\)) and those due later (i.e., \(d_{i+L-T+1}, \ldots, \ldots, \)).
If there is sufficient inventory, all this demand can be satisfied and the (positive) remaining inventory at the end of period \( i + L \) will be given by the first term above. Otherwise, the inventory level will be zero or negative and only include unsatisfied demands that were due by period \( i + L \), as given in the second term above. Replacing \( (x_i, W_i, V_i) \) with the modified inventory position \( y_i \), the above equation can be simplified to (6).

Suppose the above holds for period \( i \), it follows that

\[
x_{i+L+2} = (x_{i+L+1} + z_{i+1} - \sum_{j=i+L+1}^{i+L+T} v_j - d_{i+L+1})^+ - (x_{i+L+1} + z_{i+1} - v_{i+L+1}^+)^-
\]

\[
= (x_i + \sum_{j=1}^{i+L-1} w_j + z_i + z_{i+1} - \sum_{j=i}^{i+T-1} v_j - \sum_{j=i}^{i+L} d_j)^+ - (x_i + z_i + z_{i+1} - \sum_{j=i}^{i+L-1} v_j - \sum_{j=i}^{i+L+1-T} d_j)^-
\]

\[
= (y_{i+1} - \sum_{j=i+1}^{i+L-1} d_j)^+ - (y_{i+1} - \sum_{j=i+1}^{i+L+1-T} d_j)^-,
\]

which is consistent with (6), given the state evolution equation \( u_{i+1} = u_i + z_i - d_i \) and \( y_i = u_i + z_i \).

This completes the induction argument, and suggests that in any period, the system state is fully characterized by the modified inventory position \( y \).

We now expand the recursive equation (5) and prove the optimality of an \((s, S)\) policy with respect to the modified inventory position. Rewrite the expected discounted total cost with initial state \((x_1, W_1, V_1)\) as

\[
C_1(x_1, W_1, V_1) = \sum_{j=1}^{L} \alpha^{j-1} E_{d_1, \ldots, d_j} L(x_{j+1}) + f_1(x_1 + \sum_{j=1}^{L} w_j - \sum_{j=1}^{T} v_j^+),
\]

where

\[
f_i(u_i) = \min \{K \cdot 1_{(y_i > u_i)} + G_i(y_i)\},
\]

\[
G_i(y_i) = \alpha^L E_{d_1, \ldots, d_{i+L}} L \left( (y_i - \sum_{j=1}^{i+L} d_j)^+ - (y_i - \sum_{j=1}^{i+L-T} d_j)^- \right) + \alpha E_{d, f_{i+1}(y_i - d_i)},
\]

and \( f_{N-L+1}(\cdot) = f_{N-L+2}(\cdot) = \ldots = f_{N+1}(\cdot) = 0 \). Since no replenishment decision affects the cost incurred in periods 1 to \( L \), minimizing \( C_1(\cdot) \) is equivalent to minimizing \( f_1(\cdot) \).
Define the single-period expected cost
\[ g_i(y_i) = E_{d_i,...,d_{i+L}} L \left( (y_i - \sum_{j=i}^{i+L} d_j)^+ - (y_i - \sum_{j=i}^{i+L-T} d_j)^- \right). \]

A policy of \((s, S)\) type is optimal if \(G_i(y)\) is K-convex for all \(i = 1, \ldots, N\). We proceed by induction. Since \(G_{N-L}(y) = \alpha^T g_{N-L}(y)\) is clearly convex, it is K-convex. Then \(f_{N-L}\) is K-convex. Now suppose \(f_{i+1}\) is K-convex, then \(G_i\) is the summation of a convex function \(\alpha^T g_i(y)\) and a K-convex function \(\alpha f_{i+1}(y)\), so it is also K-convex. The remainder of the proof is similar to that in Scarf (1960), so we omit the details here.

**EC.3. Homogeneous Customer Base – Case 2: \(T > L+1\)**

**EC.3.1. Proof of Proposition 2**

Similar to Case 1, it can be derived that
\[ x_{i+L+1} = \left( x_i + \sum_{j=i}^{i+L-1} w_j + z_i - \sum_{j=i}^{i+T-1} v^j_i - \sum_{j=i}^{i+L} d_j \right)^+ - \left( x_i + \sum_{j=i}^{i+L-1} w_j + z_i - \sum_{j=i}^{i+L} v^i_j \right)^- \]

where the only difference from the previous case lies in the second parenthesis, in the calculation of the backlog. Instead of satisfying all the \(v^j_i\) for \(j = i, \ldots, i + T - 1\) in the first case (where \(i + T - 1 \leq i + L\)), now, since \(T > L + 1\), we only have to satisfy \(v^j_i, j = i, \ldots, i + L\) by the end of period \(i + L\). Satisfying the rest of the advance demands \(v^i_{i+L+1}, \ldots, v^i_{i+T-1}\) and demands \(d_i, \ldots, d_{i+L}\) by period \(i + L\) is optional. Again, substituting \((x_i, W_i, V_i)\) in the above equation with \((y_i, \hat{V}_i)\), we have (10). So the expected holding and shortage cost in period \(i + L + 1\) can be expressed as a function of the vector \((y_i, \hat{V}_i)\).

Suppose the above holds for period \(i\), it follows that
\[
\begin{align*}
x_{i+L+2} &= \left( x_{i+L+1} + z_{i+1} - \sum_{j=i+L+1}^{i+L+T} v^j_{i+L+1} - d_{i+L+1} \right)^+ - \left( x_{i+L+1} + z_{i+1} - v^i_{i+L+1} \right)^- \\
&= \left( x_i + \sum_{j=i}^{i+L-1} w_j + z_i + z_{i+1} - \sum_{j=i}^{i+T-1} v^j_i + \sum_{j=i}^{i+L} d_j \right)^+ - \left( x_i + \sum_{j=i}^{i+L-1} w_j + z_i + z_{i+1} - \sum_{j=i}^{i+L} v^i_j \right)^- \\
&= \left( y_{i+1} - \sum_{j=i+1}^{i+L+1} d_j \right)^+ - \left( y_{i+1} - \sum_{j=i+L+2}^{i+T} \hat{v}^j_{i+1} \right)^- 
\end{align*}
\]
which is consistent with (10), given the state evolution equations \( u_{i+1} = u_i + z_i - d_i \) and \( \hat{V}_{i+1} = (\hat{v}_{i+L+2}^i, \ldots, \hat{v}_{i+T-1}^i, d_i) \). This completes the induction argument, and suggests that in any period, the system state is fully characterized by \((y, \hat{V})\).

Now we can rewrite the expected discounted total cost with initial state \((x_1, W_1, V_1)\) as 
\[
C_1(x_1, W_1, V_1) = \sum_{j=1}^{L} \alpha^{j-1} E_{d_1, \ldots, d_j} L(x_{j+1}) + f_1(u_1, \hat{V}_1),
\]
where 
\[
f_i(u_i, \hat{V}_i) = \min_{y_i \geq u_i} \{ K \cdot 1_{\{y_i > u_i\}} + G_i(y_i, \hat{V}_i) \},
\]
\[
G_i(y_i, \hat{V}_i) = \alpha^L E_{d_{i}, \ldots, d_{i+L}} L \left( (y_i - \sum_{j=i}^{i+L-1} d_j)^+ - (y_i + \sum_{j=i+L+1}^{L} \hat{v}_j^i)^- \right) + \alpha E_{d_i} f_{i+1}(y_i - d_i, \hat{V}_{i+1}),
\]
and \(f_{N-L+1} = f_{N-L+2} = \cdots = f_{N+1} = 0\).

Now Proposition 2 can be proved by a similar induction argument as that in the previous case. First, \(G_{N-L}(y, \hat{V}) = \alpha^L g(y, \hat{V})\) is convex in \(y\) for any \(\hat{V}\), hence it is K-convex. Then \(f_{N-L}(u, \hat{V})\) is K-convex in \(u\) for any \(\hat{V}\). Suppose \(f_{i+1}\) is also K-convex, then \(G_i() = \alpha^L g() + \alpha f_{i+1}()\) is K-convex. Finally all \(G\) and \(f\) are K-convex, so an \((s, S)\)-type policy is optimal, but the values of the parameters \(s_i\) and \(S_i\) depend on the state vector \(\hat{V}_i\).

**EC.3.2. Proof of Proposition 3**

When \(T - L = 2\), the system state reduces to \((y_i, \hat{v}_i)\). The DP is:
\[
f_i(u_i, \hat{v}_i) = \min_{y_i \geq u_i} \{ K \cdot 1_{\{y_i > u_i\}} + G_i(y_i, \hat{v}_i) \},
\]
\[
G_i(y_i, \hat{v}_i) = \alpha^L g_i(y_i, \hat{v}_i) + \alpha E_{d_i} f_{i+1}(y_i - d_i, d_i),
\]
\[
g_i(y_i, \hat{v}_i) = E_{d_i, \ldots, d_i+L} \left[ h(y_i - d_i - \cdots - d_{i+L})^+ \right] + p(y_i + \hat{v}_i)^-,
\]
and \(f_{N+1} = 0\).

First, it is clear that \(G_N(y_N, \hat{v}_N)\) (or equivalently \(g_N\)) is decreasing in \(y_N\) when \(y_N < 0\). Suppose this also holds in period \(i + 1\), then \(f_{i+1}(u_{i+1}, \hat{v}_{i+1})\) is decreasing in \(u_{i+1}\) when \(u_{i+1} < 0\), and \(E_{d_i} f_{i+1}(y_i - d_i, d_i)\) is decreasing when \(y_i < 0\). Then \(G_i = \alpha^L g_i + \alpha E_{d_i} f_{i+1}\) is decreasing when \(y_i < 0\). This implies that \(S_i(\hat{v}_i)\), the minimizer of \(G_i\), must be larger than or equal to zero. Moreover, \(G_i\)
depends on \( \hat{v}_i \) only through the term \( p(y_i + \hat{v}_i) \). When \( y_i \geq 0 \), this term is always zero. Therefore, the minimizer \( S_i(\hat{v}_i) \) is independent of \( \hat{v}_i \).

Now consider the re-order point \( s_i(\hat{v}_i) \), defined as \( \max \{ y < S_i(\hat{v}_i) : G_i(y, \hat{v}_i) > K + G_i(S_i(\hat{v}_i), \hat{v}_i) \} \).

We know \( K + G_i(S_i(\hat{v}_i), \hat{v}_i) \) is independent of \( \hat{v}_i \), while \( G_i(y, \hat{v}_i) \) is decreasing in \( \hat{v}_i \), and is decreasing in \( y \) when \( y < S_i(\hat{v}_i) \). So \( s_i(\hat{v}_i) \) is decreasing in \( \hat{v}_i \).

### EC.4. Non-Homogeneous Customer Base – Approximation

Given the allocation assumption, it follows

\[
x_{i+L+1} = \left( x_i + \sum_{j=1}^{i+L-1} w_j + z_i - \sum_{j=1}^{i+T-1} v_j^i - \sum_{l=i}^{i+T} \sum_{j=l}^{i+T} d_l^j \right) + \left( x_i + \sum_{j=1}^{i+L-1} w_j + z_i - \sum_{j=1}^{i+L} v_j^i - \sum_{l=1}^{i+L} \sum_{j=l}^{i+L} d_l^j \right)^-.
\]

Notice that in the first parenthesis, \( \sum_{j=1}^{i+T-1} v_j^i + \sum_{l=i}^{i+T} \sum_{j=l}^{i+T} d_l^j \) are all the demands observed up to period \( i + L \), while in the second parenthesis, \( \sum_{j=1}^{i+L} v_j^i + \sum_{l=1}^{i+L} \sum_{j=l}^{i+L} d_l^j \) are those due by period \( i + L \). Clearly, \( x_{i+L+1} \) is the best possible outcome, as the allocation assumption eliminates the possible loss due to mis-allocation of on-hand units to those advance demands due later than \( i + L \).

We can apply the previous technique in EC.3.1 to collapse state \( (x_i, W_i, V_i) \) to the modified inventory position \( u_i \) and a state vector \( \hat{V}_i \) (defined in (7) and (11)), and \( \hat{v}_i^j = v_i^j \) for \( j = L+2, \ldots, T \), and simplify the above equation to (14).

The DP formulation needs to be modified to cope with vector demands:

\[
C_i(x_1, W_1, V_1) = \sum_{j=1}^{L} \alpha^{j-1} E_{D_1, \ldots, D_j} L(x_{j+1}) + f_i(u_1, \hat{V}_1),
\]

where

\[
f_i(u_i, \hat{V}_i) = \min_{y_i \geq u_i} \{ K \cdot 1_{(y_i > u_i)} + G_i(y_i, \hat{V}_i) \},
\]

\[
G_i(y_i, \hat{V}_i) = \alpha^L E_{D_1, \ldots, D_{i+L}} L(x_{i+L+1}(y_i, \hat{V}_i)) + \alpha E_{D_{i+1}} f_{i+1}(u_{i+1}, \hat{V}_{i+1}).
\]

The evolution of \((u_i, \hat{V}_i)\) follows (15), (16), and (17).

The proof of the optimality of state-dependent \((s, S)\) policy follows the same logic as that in EC.3.1, so it is omitted.
EC.5. Non-Homogeneous Customer Base – PL(Σ) Heuristic

EC.5.1. State Evolution and DP Formulation

At the beginning of period \( i \), the system state is \((x_i, v_i, v_{i+1})\). The state in period \( i + 1 \), \((x_{i+1}, v_{i+1}, v_{i+2})\), has already been derived in Equation (18) and (19). It is clear that \( x_{i+1} \) can be expressed as a function of \( y_i \) and \( \hat{v}_i \) only. To complete the induction argument, we need to show that \( x_{i+2} \) can also be expressed as the same function of \( y_{i+1} \) and \( \hat{v}_{i+1} \), where \( y_{i+1} = y_i - \sum_{j=i}^{i+2} d_i^j + z_{i+1} \) and \( \hat{v}_{i+1} = d_i^{i+2} \). After some algebraic manipulation, we have

\[
x_{i+2} = \begin{cases} 
  x_i + z_i - v_i - v_{i+1} - d_i - d_i^{i+1} + z_{i+1} - d_i^{i+1} & \text{if (1)} \\
  0 & \text{if (2)} \\
  x_i + z_i - v_i - v_{i+1} - d_i - d_i^{i+1} - d_i^{i+2} + z_{i+1} - d_i^{i+1} - d_i^{i+2} & \text{if (3)} \\
  \sigma & \text{if (4)} \\
  x_i + z_i - v_i - v_{i+1} - d_i - d_i^{i+1} - d_i^{i+2} + z_{i+1} - d_i^{i+1} - d_i^{i+2} - d_i^{i+3} & \text{if (5)} \\
  y_{i+1} + \hat{v}_{i+1} - d_i^{i+1} & \text{if } y_{i+1} + \hat{v}_{i+1} - d_i^{i+1} \leq 0; \\
  0 & \text{if } 0 \leq y_{i+1} + \hat{v}_{i+1} - d_i^{i+1} < \hat{v}_{i+1} + d_i^{i+2}; \\
  y_{i+1} - d_i^{i+1} - d_i^{i+2} & \text{if } 0 \leq y_{i+1} - d_i^{i+1} - d_i^{i+2} < \sigma; \\
  \sigma & \text{if } 0 \leq y_{i+1} - d_i^{i+1} - d_i^{i+2} - \sigma < d_i^{i+3}; \\
  y_{i+1} - \sum_{j=i}^{i+3} d_i^j & \text{if } y_{i+1} - \sum_{j=i}^{i+3} d_i^j - \sigma \geq 0.
\end{cases}
\]

The expected total discounted cost in this case is

\[
C_1(x_1, V_1) = f_1(u_1, \hat{v}_1),
\]

where

\[
f_i(u_i, \hat{v}_i) = \min_{y_i \geq u_i} \{K \cdot 1_{(y_i > u_i)} + G_i(y_i, \hat{v}_i)\},
\]

\[
G_i(y_i, \hat{v}_i) = E_{D_i} L(x_{i+1}(y_i, \hat{v}_i)) + \alpha E_{D_{i+1}} f_{i+1}(y_i - \sum_{j=i}^{i+2} d_i^j, d_i^{i+2}),
\]

and \( f_{N+1}(\cdot) = 0 \).

The inventory holding and backorder cost \( L(x_{i+1}(y_i, \hat{v}_i)) \) is no longer convex, so the previous proof technique based on \( K \)-convexity is not applicable. In the following proof, we show that if the probability density functions (or mass functions for discrete demand) of the demand are logconcave (or strongly unimodal), then the functions \( G_i(\cdot, \hat{v}), 1 \leq i \leq N \), are \( K \)-quasiconvex, and the optimal policy is still a state-dependent \((s, S)\) policy.
EC.5.2. Proof of the Optimality of State-dependent \((s, S)\) Policy

Using the result of Lemma EC.3, we first show that the single-period expected cost is quasiconvex.

**Lemma EC.6.** The single period expected cost \(g(y_i, \hat{v}_i) = E_{D_i}[L(x_{i+1}(y_i, \hat{v}_i))]\) is quasiconvex in \(y_i\), for any given \(\hat{v}_i\).

**Proof.** The cost \(L()\) can be separated into three parts, i.e.,

\[
L(x_{i+1}(y_i, \hat{v}_i)) = \psi_0(y_i, \hat{v}_i) + \psi_1(y_i) + \psi_2(y_i),
\]

where

\[
\psi_0(y_i, \hat{v}_i) = \begin{cases} 
-p[y_i + \hat{v}_i - d_i^t], & \text{if } y_i + \hat{v}_i - d_i^t \leq 0; \\
0, & \text{if } y_i + \hat{v}_i - d_i^t > 0,
\end{cases}
\]

\[
\psi_1(y_i) = \begin{cases} 
0, & \text{if } y_i - d_i^t - d_i^{t+1} < 0; \\
h[y_i - d_i^t - d_i^{t+1}], & \text{if } 0 \leq y_i - d_i^t - d_i^{t+1} < \sigma; \\
h\sigma, & \text{if } y_i - d_i^t - d_i^{t+1} \geq \sigma,
\end{cases}
\]

and

\[
\psi_2(y_i) = \begin{cases} 
0, & \text{if } y_i - \sum_{j=i}^{i+2} d_j^t - \sigma < 0; \\
h[y_i - \sum_{j=i}^{i+2} d_j^t], & \text{if } y_i - \sum_{j=i}^{i+2} d_j^t - \sigma \geq 0.
\end{cases}
\]

In words, \(\psi_0()\) is the penalty cost associated with existing backorders and unsatisfied demand \(d_i^t\), which is decreasing in \(y_i\). \(\psi_1()\) is the holding cost incurred by the inventory used as safety stock, while \(\psi_2()\) is the holding cost incurred by the excess inventory after satisfying all the demands. Clearly \(\psi_0(y_i, \hat{v}_i)\) is decreasing in \(y_i\), and \(\psi_1()\) and \(\psi_2()\) are increasing in \(y_i\).

Assuming \(d_i^t\) is given, the conditional expectation of \(L()\) with respect to \(d_i^{t+1}\) and \(d_i^{t+2}\) is

\[
E_{d_i^{t+1},d_i^{t+2}|d_i^t}[L()] = \psi_0() + E_{d_i^{t+1},d_i^{t+2}|d_i^t}[\psi_1() + \psi_2()].
\]

This conditional expectation is quasiconvex in \(y_i - d_i^t\), since the first term is decreasing in \(y_i - d_i^t\) when \(y_i - d_i^t \in (-\infty, -\hat{v}_i]\) and is equal to zero on the interval \([-\hat{v}_i, \infty)\), and the second term is equal to zero when \(y_i - d_i^t \in (-\infty, 0]\) and is increasing on \([0, \infty)\) (monotonicity is preserved through expectation). Moreover, if \(d_i^t\) is logconcavely distributed, we know that by Lemma EC.3, \(g(y_i, \hat{v}_i) = E_{D_i}[L(y_i, \hat{v}_i)] = E_{d_i^t} E_{d_i^{t+1},d_i^{t+2}|d_i^t}[L(y_i, \hat{v}_i)]\) is quasiconvex in \(y_i\). \qed
Lemma EC.7. The global minimizer of $g(\cdot, v)$, defined as

$$y_0(v) = \arg \min_{y \in \mathbb{Z}} g(y, v),$$

is decreasing in $v$, and $y_0(v) - y_0(v+1) \leq 1$.

Proof. Although $y$ is discrete in our model, here we prove the lemma by assuming it is continuous. Write the first-order condition,

$$H(y, v) = \frac{\partial g(y, v)}{\partial y} = h\Phi^{012}(y - \sigma) + h(\Phi^{01}(y) - \Phi^{01}(y - \sigma)) + p(\Phi^{\sigma}(y + v) - 1) = 0,$$

where $\Phi^{012}$, $\Phi^{01}$, and $\Phi^{\sigma}$ ($\phi^{012}$, $\phi^{01}$, and $\phi^{\sigma}$) are the probability distribution (density) functions of $d^i + d^{i+1} + d^{i+2}$, $d^i$, and $d^i$, respectively. By definition, $(y_0(v), v)$ satisfies the above condition, i.e., $H(y_0(v), v) = 0$. By the Implicit Function Theorem, we have

$$\frac{dy_0(v)}{dv} = -\frac{\partial H}{\partial y_0} = -\frac{p\phi^{\sigma}(y_0 + v)}{h\phi^{012}(y_0 - \sigma) + h(\phi^{01}(y_0) - \phi^{01}(y_0 - \sigma)) + p\phi^{\sigma}(y_0 + v)}.$$ 

Since $y_0(v)$ is the global minimizer of $g(\cdot, v)$, $g(\cdot, v)$ is locally convex at $y_0(v)$ and the denominator, which is the second-order derivative of $g(\cdot, v)$ evaluated at $y_0(v)$, is positive. Thus $\frac{dy_0(v)}{dv} \leq 0$. Also, since $g(\cdot, v)$ is quasiconvex, $H(\cdot, v)$ goes from negative to positive and crosses zero only once. The first and the third terms of $H$ are increasing, while the second term is increasing then decreasing (due to the logconcavity or strong unimodality of $\phi^{01}$). So $h\Phi^{012}(y - \sigma) + h(\Phi^{01}(y) - \Phi^{01}(y - \sigma))$ is either increasing or increasing-decreasing-increasing. If it is increasing, then $H$ is increasing, which implies $g$ is convex and the above derivation is not necessary. If it is increasing-decreasing-increasing, then at the zero-crossing point $(y_0, v)$, the term $h\Phi^{012}(y - \sigma) + h(\Phi^{01}(y) - \Phi^{01}(y - \sigma))$ must be increasing, or equivalently, $h\phi^{012}(y_0 - \sigma) + h(\phi^{01}(y_0) - \phi^{01}(y_0 - \sigma)) \geq 0$. Therefore $\frac{dy_0(v)}{dv} \geq -1$. □

Finally we are ready to prove the optimal policy is $(s, S)$ type. The proof is a similar induction argument as what we have done in the previous section, however, now we need to show that the $G_k$ and $f_k$ are $K$-quasiconvex instead of $K$-convex.
First, since \( f_{N+1}(\cdot) = 0 \), by Lemma EC.6, we have \( G_N(\cdot, \hat{v}) \) is quasiconvex and hence \( K \)-quasiconvex. Then, by definition, \( G_N(\cdot, \hat{v}) \) is decreasing on \( (-\infty, y_0(\hat{v})) \) and \( K \)-increasing on \([y_0(\hat{v}), \infty)\).

Now suppose \( G_{i+1}(\cdot, \hat{v}), \ 1 \leq i + 1 \leq N \), is decreasing on \( (-\infty, y_0(\hat{v})) \) and \( K \)-increasing on \([y_0(\hat{v}), \infty)\). Then \( f_{i+1}(u, \hat{v}) = \min_{y > u} [K \cdot 1_{\{y > u\}} + G_{i+1}(y, \hat{v})] \) is \( K \)-increasing for \( u \in \mathbb{R} \). Moreover, for any \( u_2 < u_1 < y_0(\hat{v}) \),

\[
f_{i+1}(u_2, \hat{v}) = \min\{G_{i+1}(u_2, \hat{v}), K + G_{i+1}(y^*(\hat{v}), \hat{v})\}
\]

\[
\geq \min\{G_{i+1}(u_1, \hat{v}), K + G_{i+1}(y^*(\hat{v}), \hat{v})\}
\]

\[
= f_{i+1}(u_1, \hat{v}),
\]

which implies \( f_{i+1}(\cdot, \hat{v}) \) is decreasing on \( (-\infty, y_0(\hat{v})) \).

To complete the induction, we want to show that

\[
G_i(y_i, \hat{v}_i) = g_i(y_i, \hat{v}_i) + \alpha E_{D_i}[f_{i+1}(y_i - \sum_{j=i}^{i+2} d_j^i, d_j^{i+2})]
\]

is decreasing on \( (\infty, y_0(\hat{v}_i)) \) and \( K \)-increasing on \([y_0(\hat{v}_i), \infty)\), and therefore is \( K \)-quasiconvex (Lemma EC.5). \( g_i(\cdot, \hat{v}_i) \) is quasiconvex with global minimum \( y_0(\hat{v}_i) \). The expectation term is \( K \)-increasing, as the property is preserved through expectation (Lemma EC.4). The remaining element is to show that the expectation term is decreasing on \( y < y_0(\hat{v}_i) \).

It has been shown that \( f_{i+1}(u, \hat{v}) \) is decreasing on \( (-\infty, y_0(\hat{v})) \). Then for any given \( D_i = (d_i^1, d_i^{i+1}, d_i^{i+2}) \), \( f_{i+1}(y_i - \sum_{j=i}^{i+2} d_j^i, d_j^{i+2}) \) is decreasing when \( y_i \in (-\infty, y_0(d_i^{i+2}) + \sum_{j=i}^{i+2} d_j^i) \). If \( d_i^{i+2} \leq \hat{v}_i \), then \( y_0(d_i^{i+2}) \geq y_0(\hat{v}_i) \) and hence \( y_0(d_i^{i+2}) + \sum_{j=i}^{i+2} d_j^i \geq y_0(\hat{v}_i) \). If \( d_i^{i+2} > \hat{v}_i \), by Lemma EC.7, \( y_0(d_i^{i+2}) - y_0(\hat{v}_i) \geq \hat{v}_i - d_i^{i+2} \). Then

\[
y_0(d_i^{i+2}) - y_0(\hat{v}_i) + \sum_{j=i}^{i+2} d_j^i \geq \hat{v}_i - d_i^{i+2} + \sum_{j=i}^{i+2} d_j^i = \hat{v}_i + \sum_{j=i}^{i+4} d_j^i \geq 0.
\]

Therefore for any \( D_i \), \( f_{i+1}(y_i - \sum_{j=i}^{i+2} d_j^i, d_j^{i+2}) \) is decreasing when \( y < y_0(\hat{v}_i) \). Thus, its expectation is also decreasing on \( (-\infty, y_0(\hat{v}_i)) \), which completes the proof.
References


