Design of a Bi-Metallic Strip for a Thermal Switch

*Team Design Project 2*

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ABSTRACT

The goal of this project was to design a bi-metallic strip so that the free end of the strip would deflect 0.05 inches when there is a 50°F change in temperature relative to its reference temperature. The strip had to be 3 inches long with one end fixed, and both metals had the same width, b. We were allowed to freely choose both materials as long as they were metallic. The deflection could be controlled by changing their materials (which changes the coefficient of thermal expansion and the elastic modulus) and the height of the material. The first step we took to finding the two height values was to set the two deflection values equal to each other and find the force in the bi-metallic switch. We then used this force value to then calculate the moments in the bi-metallic switch. Next we used the integration method to find the beam deflection for this problem. After this we had to take our composite strip and turn it into one strip made of only one material that still has the same properties as the composite but first find the center of mass of the composite. Using this information, we then found the second moment of area for the composite strip. We plugged the moment equation and the equation for the second moment of area into the beam deflection equation into the equation for maximum deflection. By choosing two materials and the height for one, we could solve for the height of the other which gave us the dimensions for our strip.

INTRODUCTION

A bi-metallic strip is a composite beam composed of two materials with different coefficients of thermal expansion. As the temperature changes the two components of the bi-metallic strip will elongate or contract by different lengths causing the strip to bend since both components are bonded together. The strip will bend a specific way if it is heated and will bend the opposite if it is cooled relative to its reference temperature which can be seen in Figure 1. There are many uses for a bi-metallic strip such as creating an inexpensive thermostat since they either open or close an electrical contact at a predetermined temperature. The goal of this project was to design a bi-metallic strip for this sort of application that would deflect 0.05 inches when there is a 50°F change in temperature relative to its reference temperature. The other constraints included that the strip had to be 3 inches long with one end fixed as well as that both strips had to have the same width. We were allowed to choose any two materials as long as they were both metallic. The deflection value and the calculated height values were used to assess the success of the design. We hypothesized that as you increase the height, the deflection will decrease. Because of this, we will need to use small h-values for both material 1 and material 2.

Figure 1
ANALYSIS & DESIGN

As heat is added, one of the metals will expand faster than the other leading to a force equal in magnitude but opposite in direction acting on both metals. This means that the change in length of each metal strip depends on the thermal expansion as well as the deformation from axial loading. We also know that these deformations will be equal to one another since the strips are permanently attached. We can solve for the force acting on these strips in terms of material and physical properties of the metals.

\[ \delta_1 = \delta_2 \]

\[ -\frac{PL}{E_1A_1} + \alpha_1 L \Delta T = \frac{PL}{E_2A_2} + \alpha_2 L \Delta T \]

\[ P \left( \frac{1}{E_1A_1} + \frac{1}{E_2A_2} \right) = \Delta T (\alpha_1 - \alpha_2) \]

\[ P \left( \frac{E_1A_1 + E_2A_2}{E_1A_1E_2A_2} \right) = \Delta T (\alpha_1 - \alpha_2) \]

\[ P = \frac{\Delta T (\alpha_1 - \alpha_2)(E_1A_1E_2A_2)}{E_1A_1 + E_2A_2} \]

\[ P = \frac{\Delta T (\alpha_1 - \alpha_2)(E_1bh_1E_2bh_2)}{E_1bh_1 + E_2bh_2} \]
\[ p = \frac{\Delta T(\alpha_1 - \alpha_2)b(E_1h_1E_2h_2)}{E_1h_1 + E_2h_2} \]

The moment that acts on the strip is equivalently written as acting in the middle of the height. Since this is the case, the moment equation can be expressed as the force that is experienced times the average height which is the middle of the beam.

\[ M = p \left( \frac{h_1 + h_2}{2} \right) \]

\[ M = \frac{\Delta T(\alpha_1 - \alpha_2)b(E_1h_1E_2h_2)(h_1 + h_2)}{2(E_1h_1 + E_2h_2)} \]

Now, by using the integration method, we can find an equation that represents the deflection in the \( y \)-direction of the bi-metallic strip. First we must represent the moment at any point of the beam as a function of \( x \).

\[ \sum F_y = R = 0 \]
\[
\sum M_o = M_o - M = 0 \Rightarrow M_o = M
\]

\[
\sum M_o - M(x) = 0 \Rightarrow M(x) = M
\]

Figure 5

Now that we have our moment equation, we can find our equation for beam deflection.

\[ Ely'' = M(x) \]

\[ Ely'' = M \]

\[ Ely' = Mx + C_1 \]

The slope at the support of a cantilever beam is equal to zero; therefore,

\[ Ely'|_{x=0} = 0 = C_1 \]

\[ Ely = \frac{1}{2} Mx^2 + C_2 \]

The deflection at the support of a cantilever beam is equal to zero; therefore,

\[ Ely|_{x=0} = 0 = C_2 \]

\[ y = \frac{1}{2} \frac{Mx^2}{EI} \]

Now that we have our general equation for the deflection at any point on the strip, we can plug in \( L \) for \( x \) since we want to find the deflection at the end of the strip.

\[ y|_{x=L} = \frac{1}{2} \frac{ML^2}{EI} \]

\[ \kappa = \frac{M}{EI} \Rightarrow y = \frac{\kappa L^2}{2} \]

We must now take our composite strip and turn it into one strip made of only one material that still has the same properties as the composite. The all-material one transformed section is obtained by scaling every horizontal dimension in the material two part of the composite strip by the ratio \( n = \frac{E_2}{E_1} \). We must first find the center of mass of the composite.
\[\begin{aligned}
\gamma_{c1} &= h_2 + \frac{1}{2} h_1 \\
A_1 &= bh_1 \\
\gamma_{c2} &= \frac{1}{2} h_2 \\
A_2 &= bn h_2
\end{aligned}\]

\[
\bar{y}_c = \frac{1}{A_T} \sum_{i=1}^{n} y_c A_i = \frac{1}{b(h_1 + nh_2)} [bh_1 h_2 + \frac{1}{2} bh_1^2 + \frac{1}{2} bn h_2^2] = \frac{1}{h_1 + nh_2} [h_1 h_2 + \frac{1}{2} h_1^2 + \frac{1}{2} nh_2^2]
\]

Now that we have an equation that represents our center of mass, we must find the second moment of area for the composite strip.

\[
I_1 = \frac{1}{12} bh_1^3 + \gamma_{c1}^2 A_1 = \frac{1}{12} bh_1^3 + \left(\frac{h_2 + \frac{1}{2} h_1}{\bar{y}_c} - \bar{y}_c\right)^2 (bh_1)
\]

\[
I_2 = \frac{1}{12} bh_2^3 + \gamma_{c2}^2 A_2 = \frac{1}{12} nb h_2^3 + \left(\frac{1}{2} h_2 - \bar{y}_c\right)^2 (nb h_2)
\]

\[
I = I_1 + I_2
\]

\[
I = \frac{1}{12} bh_1^3 + \left(h_2 + \frac{1}{2} h_1\right)^2 (bh_1) + \frac{1}{12} nb h_2^3 + \left(\frac{1}{2} h_2\right)^2 (nb h_2)
\]

\[
I = \frac{1}{12} bh_1^3 + \left(\frac{h_2 + \frac{1}{2} h_1}{\bar{y}_c} + \bar{y}_c^2 - 2\bar{y}_c \left(h_2 + \frac{1}{2} h_1\right)\right) (bh_1) + \frac{1}{12} nb h_2^3
\]

\[
+ \left[\left(\frac{1}{2} h_2\right)^2 + \bar{y}_c^2 - 2\bar{y}_c \left(\frac{1}{2} h_2\right)\right] (nb h_2)
\]

\[
I = \frac{1}{12} bh_1^3 + \left[h_2^2 + \frac{1}{4} h_1^2 + h_1 h_2 + \bar{y}_c^2 - 2\bar{y}_c h_2 - \bar{y}_c h_1\right] (bh_1) + \frac{1}{12} nb h_2^3
\]

\[
+ \left[\frac{1}{4} h_2^2 + \bar{y}_c^2 - \bar{y}_c h_2\right] (nb h_2)
\]

\[
I = \frac{1}{12} bh_1^3 + bh_1 h_2^2 + \frac{1}{4} bh_1^3 + bh_1^2 h_2 + bh_1 \bar{y}_c^2 - 2b\bar{y}_c h_1 h_2 - b\bar{y}_c h_2^2 + \frac{1}{12} nb h_2^3 + \frac{1}{4} nb h_2^3
\]

\[
+ nb \bar{y}_c^2 h_2 - nb \bar{y}_c h_2^2
\]

\[
I = b \left[\frac{1}{3} h_1^3 + h_1 h_2^2 + h_1^2 h_2 + \bar{y}_c^2 h_1 - 2\bar{y}_c h_1 h_2 - \bar{y}_c h_1^2 + \frac{1}{3} nh_2^3 + \bar{y}_c^2 nh_2 - \bar{y}_c nh_2^2\right]
\]
We can now plug in the equations we derived for M and I into our maximum deflection equation.

\[
\gamma_{\text{max}} = \frac{1}{2} \frac{ML^2}{EI}
\]

\[
\gamma_{\text{max}} = \frac{L^2 \Delta T (\alpha_1 - \alpha_2)(E_1 h_1 E_2 h_2)(h_1 + h_2)}{4(E_1 h_1 + E_2 h_2)E_1 \left( \frac{1}{3} h_1^3 + h_1 h_2^2 + h_1^2 h_2 + \frac{\bar{\gamma}_c}{2} h_1 - 2 \bar{\gamma}_c h_1 h_2 - \frac{\bar{\gamma}_c}{2} h_1^2 + \frac{1}{3} n h_2^3 + \frac{\bar{\gamma}_c}{2} n h_2 - \frac{\bar{\gamma}_c}{2} n h_2^2 \right)}
\]

Where:

\[
\bar{\gamma}_c = \frac{1}{h_1 + nh_2} \left[ h_1 h_2 + \frac{1}{2} h_1^2 + \frac{1}{2} nh_2^2 \right]
\]

\[
n = \frac{E_2}{E_1}
\]

We know that the maximum deflection of the 3 inch (0.0762 m) long strip at a temperature change of 50°F (27.78°C) must be 0.05 in (0.00127 m). In our design, we are free to pick the materials, the heights of both strips of materials, and the base length. Most bi-metallic strips are made out of steel and copper, so we will use these as our materials. The thermal expansion of the copper is higher than that of the steel, so, in our picture, the copper is the metal on the top, and the steel is the metal on the bottom. The strip will deflect downward since the copper expands more than the steel when heated. When we created our drawing for the composite beam, we decided that E\text{2} was greater than E\text{1}, so we will consider anything with subscript 1 as the properties of copper and anything with subscript 2 as the properties for steel. The height for the copper, h\text{1}, was arbitrarily chosen to be 0.0001 m, and our equation for maximum deflection was solved for the height of the steel, h\text{2}.

<table>
<thead>
<tr>
<th>Material</th>
<th>Elastic Modulus (GPa)</th>
<th>Coefficient of Thermal Expansion (10^{-6}/°C)</th>
<th>Height (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper</td>
<td>117</td>
<td>16.6</td>
<td>0.0001</td>
</tr>
<tr>
<td>Steel</td>
<td>200</td>
<td>11.9</td>
<td>?</td>
</tr>
</tbody>
</table>

For the ease of our calculations, we used the simplified version of our formula which was found on http://en.wikipedia.org/wiki/Bimetallic_strip. We validated their equation with ours with dummy numbers for our variables of E\text{1}, E\text{2}, h\text{1}, and h\text{2} and compared the result of our maximum deflection with that of ours. Their equation for maximum deflection is as follows:

\[
\gamma_{\text{max}} = \frac{3L^2 \Delta T (\alpha_1 - \alpha_2)(E_1 h_1 E_2 h_2)(h_1 + h_2)}{E_1^2 h_1^4 + 4E_1 E_2 h_1^3 h_2 + 6E_1 E_2 h_1^2 h_2^2 + 4E_1 E_2 h_1 h_2^3 + E_2^2 h_2^4}
\]

0.00127 m

\[
3(0.0762 m)^2(27.78°C) \left[ (16.6 - 11.9) \cdot 10^{-6} \right] \left( (117 \text{ GPa})(0.0001 m)(200 \text{ GPa})h_2 (0.0001 m + h_2) \right)
\]

\[= \left( (117 \text{ GPa})^2 (0.0001 m)^2 + 4(117 \text{ GPa})(200 \text{ GPa})(0.0001 m)^3 h_2 + 6(117 \text{ GPa})(200 \text{ GPa})(0.0001 m)^3 h_2^2 + 4(117 \text{ GPa})(200 \text{ GPa})(0.0001 m)^3 h_2^3 + (200 \text{ GPa})^3 h_2^4 \right)
\]

0.00127 m

\[
\frac{5.322 \cdot 10^{12} h_2 (0.0001 + h_2)}{1.3689 \cdot 10^6 + 93.6 \cdot 10^9 h_2 + 1.404 \cdot 10^{15} h_2^2 + 9.36 \cdot 10^{18} h_2^3 + 40 \cdot 10^{21} h_2^4}
\]

\[h_2 = -0.000363 \text{ or } -0.00010192 \text{ or } 0.0000040665 \text{ or } 0.000227233 \text{ m}
\]
Two of these answers are negative and, therefore, illogical, and one value is too small to be applicable, so our height for $h_2$ is 0.0002272 m.

**RESULTS**

Our design for a bi-metallic strip uses the materials copper and steel. One of the possible combinations of the height dimensions is as follows: the height of the copper is 0.0001 meters (0.003937 inches), and the height of the steel is 0.000227 meters (0.008962 inches). The base could be any dimension of our choosing since it canceled out of our deflection formula, so it can essentially be any length of our choosing. The only condition is that it is larger than the total height so that the beam deflects downward instead of from side to side. We will set our base as 0.005 meters (0.19685 inches).

<table>
<thead>
<tr>
<th>Material</th>
<th>Height (m)</th>
<th>Height (in)</th>
<th>Base (m)</th>
<th>Base (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper</td>
<td>0.0001</td>
<td>0.003937</td>
<td>0.005</td>
<td>0.19685</td>
</tr>
<tr>
<td>Steel</td>
<td>0.000227</td>
<td>0.008962</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
</tbody>
</table>

![Diagram](image)

Figure 7

Some shortcomings with this design are that the heights required are so small and require such fine tolerances that they will likely be expensive to produce. With just a one percent difference in heights between the two materials, the maximum deflection experienced with a 50°F temperature change will change from 0.00127 meters to 0.0012506 meters. This is a 1.527% difference in deflection while machining with a ±0.000001m tolerance which would be unbelievably expensive. This could be accounted for by machining with a lower and less expensive tolerance, testing the deflection, and then adjusting the length to what it would need to be in order to achieve a 0.00127 meter deflection.