Analysis of Gain Scheduled Control for Nonlinear Plants

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Abstract—Gain scheduling has proven to be a successful design methodology in many engineering applications. However, in the absence of a sound theoretical analysis, these designs come with no guarantees on the robustness, performance, or even nominal stability of the overall gain scheduled design.

This paper presents such an analysis for two types of nonlinear gain scheduled control systems: 1) scheduling on a reference trajectory; and 2) scheduling on the plant output. Conditions are given which guarantee stability, robustness, and performance properties of the global gain scheduled designs. These conditions confirm and formalize popular notions regarding gain scheduled designs, such as the scheduling variable should "vary slowly" and "capture the plant's nonlinearities." These results extend previous work by the authors which addressed the case of linear plants whose dynamics depend on exogenous parameters.

I. INTRODUCTION

Gain scheduling (e.g., [9]) has proven to be a successful design methodology in many engineering applications. However, in the absence of a sound analysis, these designs come with no guarantees on the robustness, performance, or even nominal stability of the overall gain scheduled design.

In earlier papers [7], [8], this issue of guaranteed properties was addressed for one class of gain scheduled control systems, namely, linear parameter-varying plants. This paper continues this discussion of guaranteed properties with the focus of attention now on nonlinear plants.

A typical gain scheduled design procedure for nonlinear plants is as follows.

1) The designer selects several operating points which cover the range of the plant's dynamics. In contrast to the linear parameter-varying case, these operating points are usually indexed by some combination of state variables or reference state trajectories.

2) At each of these operating points, the designer constructs a linear time-invariant approximation to the plant and designs a linear compensator for each linearized plant.

3) In between operating points, the parameters (gains) of the compensators are then interpolated, or scheduled, thus resulting in a global compensator.

Since the local designs are based on linear time-invariant approximations to the plant, the designer can guarantee that at each operating point, the feedback system has the needed feedback properties, such as robust stability, robust performance, and of course nominal stability. Since the actual system is nonlinear, the overall gain scheduled system need not have any of these properties—even nominal stability. In other words, one typically cannot assess a priori the guaranteed stability, robustness, and performance properties of gain scheduled designs. Rather, any such properties are inferred from extensive computer simulations.

In addition to simulations, gain scheduled designs are guided by heuristic rules-of-thumb. The two most fundamental guidelines are: 1) the scheduling variable should vary slowly; and 2) the scheduling variable should capture the plant's nonlinearities. As in the linear parameter-varying case, these guidelines are simply reminders that the local operating point designs were based on linear time-invariant approximations to the actual plant. Thus, these approximations must be sufficiently accurate if one expects the local feedback properties to carry over to the overall gain scheduled system.

In this paper, two types of nonlinear gain scheduled systems are analyzed: 1) a nonlinear plant scheduling on a reference trajectory; and 2) a nonlinear plant scheduling on the plant output. In each case, sufficient conditions are given which guarantee that the overall gain scheduled system will retain the feedback properties of the local designs. These conditions formalize the rules-of-thumb which have guided successful gain scheduled designs. Again, the most fundamental idea behind the analysis is that the original designs are based on linear time-invariant approximations of a nonlinear plant.

The remainder of this paper is organized as follows. In Section II, the notation and some mathematical preliminaries are given. Furthermore, a brief review of the results from [7], [8] regarding stability of linear time-varying Volterra integrodifferential equations (VIDE's) is presented. Section III addresses the first of two nonlinear gain scheduled situations, scheduling of a reference trajectory. The formal problem statement is given in Section III-A. In Section III-B, conditions are given which guarantee the stability, robustness, and performance of the overall gain scheduled design. Section IV presents the second situation, scheduling on the plant output. Section IV-A explains the output scheduling design process. In Section IV-B, conditions are given which guarantee that the robust stability and robust performance properties of the local designs carry over to the overall design. Finally, concluding remarks are given in Section V. Throughout this paper, the tools developed in [7], [8] form the backbone of the analysis.

II. BACKGROUND MATERIAL

A. Notation and Mathematical Preliminaries

First, some notation regarding standard concepts for analysis of feedback systems (e.g., [3], [12]) is established.

\( \mathbb{R} \) denotes the field of real numbers, \( \mathbb{R}^+ \) the set \( \{ t \in \mathbb{R} | t \geq 0 \} \), \( \mathbb{R}^n \) the set of \( n \times 1 \) vectors with elements in \( \mathbb{R} \), and \( \mathbb{R}^{n \times m} \) the set of \( n \times m \) matrices with elements in \( \mathbb{R} \). \( A_{ij} \) denotes the \( ij \)-th element of the matrix \( A \in \mathbb{R}^{n \times m} \), \( | \cdot | \) denotes both the vector norm on \( \mathbb{R}^n \) and its induced matrix norm on \( \mathbb{R}^{n \times m} \). \( \text{D}f \) denotes the derivative of \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \). \( \text{D}_f \) denotes the derivative with respect to the \( h \)-th variable of \( f : \mathbb{R}^{n+1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_i} \times \cdots \times \mathbb{R}^{n_k} \rightarrow \mathbb{R}^m \).

Let \( f : \mathbb{R} \rightarrow \mathbb{R} \). \( \text{D}f \) denotes the Dini derivative of \( f \) defined by

\[ \text{D}^+ f(x_0) = \limsup_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \]

Let \( f : \mathbb{R} + \rightarrow \mathbb{R}^m \). \( \hat{f} \) denotes the Laplace transform of \( f \), \( \hat{f} \).
denotes the truncation operator on $f$ defined by

$$
\vartheta_T f(t) = \begin{cases} 
  f(t), & t \leq T; \\
  0, & t > T.
\end{cases}
$$

$\mathcal{W}_{T,e}$ denotes the truncation and exponential weighting operator on $f$ defined by

$$
\mathcal{W}_{T,e} f(t) = \begin{cases} 
  e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e^{-e{-}}}\\n
\end{cases}
\end{align}

$\mathcal{L}_\infty$ denotes the standard Lebesgue function space of measurable essentially bounded functions. $\mathfrak{B}$ denotes the set of measurable functions $f : \mathbb{R} \to \mathbb{R}^n$ such that

$$
\|f\|_{\mathfrak{B}} = \sup_{t \in \mathbb{R}} |f(t)| < \infty.
$$

$\mathfrak{A}(\sigma)$ denotes the set whose elements are of the form

$$
f(t) = \left\{ \begin{array}{ll}
  f_\sigma(t) + \sum_{i=0}^{\infty} f_i \delta(t-t_i), & t \geq 0; \\
  0, & t < 0
\end{array} \right.
$$

where $f_\sigma : \mathbb{R} \to \mathbb{R}, \ t_i \geq 0, f_i \in \mathfrak{B}$, and

$$
\|f\|_{\mathfrak{A}(\sigma)} = \sum_{i=0}^{\infty} |f_i| < \infty.
$$

For any two elements $f$ and $g$ of $\mathfrak{A}(\sigma), f+g$ denotes the convolution of $f$ and $g$. $\mathfrak{A}^{n,m}(\sigma)$ denotes the set of $n \times m$ matrices whose elements are in $\mathfrak{A}(\sigma)$. Let $\hat{\Delta} \in \mathfrak{A}^{n \times m}(\sigma)$ and let $\Delta = \hat{\Delta}^t \in \mathfrak{A}^{m \times n}$ be defined as $\Delta = \| \Delta \|_{\mathfrak{A}(\sigma)}$. Then $\| \Delta \|_{\mathfrak{A}(\sigma)} = \| \Delta \|_{\mathfrak{A}(\sigma)}$. Finally, $\hat{\Delta}(\sigma)$ and $\Delta^{m \times n}(\sigma)$ are defined as the set of Laplace transforms of elements of $\mathfrak{A}(\sigma)$ and $\mathfrak{A}^{m \times n}(\sigma)$, respectively. For further details on $\mathfrak{A}(\sigma)$ and $\mathfrak{A}^{m \times n}(\sigma)$, see [2], [3].

B. Linear Volterra Integrodifferential Equations

This section summarizes the main results from [7], [8] for equations of the form

$$\dot{x}(t) = A(t) x(t) + \int_0^t B(t) \Delta(t - \tau) C(t) x(\tau) d\tau, \quad t \geq \tau \tag{1}$$

with initial condition

$$x(\theta) = \phi(\theta), \quad 0 \leq \theta \leq \theta_0, \phi \in \mathfrak{B};$$

$$x(t_0) = \phi(t_0), \tag{2}$$

where it is assumed that $x(t) \in \mathfrak{A}^n$ and $\Delta \in \mathfrak{A}^{m \times p}(\sigma)$ for some $\sigma \geq 0$.

The following assumption is made on (1).

Assumption 2.1: The matrices $A$, $B$, and $C$ are bounded and globally Lipschitz continuous. That is, there exist constants $k_A, k_B, k_C \geq 0$ and $L_A, L_B, L_C \geq 0$ such that $\forall t, \tau \in \mathbb{R}$,

$$\|A(t)\| \leq k_A, \quad \|A(t) - A(\tau)\| \leq L_A |t - \tau|,$$

$$\|B(t)\| \leq k_B, \quad \|B(t) - B(\tau)\| \leq L_B |t - \tau|,$$

$$\|C(t)\| \leq k_C, \quad \|C(t) - C(\tau)\| \leq L_C |t - \tau|.$$

A definition of exponential stability is now presented.

Definition 2.1: The VIDE (1) with initial condition (2) is said to be exponentially stable if there exist constants $m, \lambda, \beta > 0$, where $\beta \geq \lambda$ such that for $t \geq t_0$,

$$\|x(t)\| \leq me^{-\lambda(t-t_0)} \|\mathcal{W}_{t_0, \phi}\|_{\mathfrak{B}}.$$

It is stressed that the constants $m$, $\lambda$, and $\beta$ are independent of the initial condition $\phi(t_0)$. The convention $\beta \geq \lambda$ follows from the reasoning that solutions to (1) cannot decay faster that they are forgotten. Furthermore, this inequality will be needed in subsequent proofs (e.g., Theorem 3.2).

In the case of time-invariant $A$, $B$, and $C$ matrices, one has the following condition for exponential stability.

Theorem 2.1 [7], [8]: Consider the time-invariant VIDE

$$\dot{x}(t) = Ax(t) + \int_0^t B \Delta(t - \tau) C x(\tau) d\tau, \quad t \geq \tau \tag{2}$$

with initial condition (2). A sufficient condition for exponential stability is that there exists a constant $\beta > 0$ such that

$$s \rightarrow (sl - A - B \Delta(s) C)^{-1} \in \hat{\mathfrak{A}}^{n \times m}(-2\beta),$$

$$\hat{\Delta} \in \hat{\mathfrak{A}}^{m \times p}(-2\beta).$$

Finally, the following theorem gives a sufficient condition for exponential stability of (1) in the case where (1) is exponentially stable for all frozen values of $s$. This generalizes a standard result for ordinary differential equations (e.g., [9]).

Theorem 2.2 [7], [8]: Consider the VIDE (1) with initial condition (2) under Assumption 2.1. Assume that there exists a constant $\beta > 0$ such that

$$s \rightarrow (sl - A(\tau) - B(\tau) \Delta(s) C(\tau))^{-1} \in \hat{\mathfrak{A}}^{n \times m}(-2\beta), \quad \forall \tau \in \mathbb{R},$$

$$\hat{\Delta} \in \hat{\mathfrak{A}}^{m \times p}(-2\beta).$$

Finally, define the following measure of time-variations of (1):

$$K \triangleq L_A + L_B \|\Delta\|_{\mathfrak{A}(\sigma)} k_C + k_B \|\Delta\|_{\mathfrak{A}(\sigma)} L_C. \tag{3}$$

Under these conditions, the VIDE (1) is exponentially stable for sufficiently small $K$, or equivalently, for sufficiently slow time-variations (see [7], [8] for explicit formulas).

It is noted that the hypotheses of Theorem 2.2 imply exponential stability of (1) for all frozen values of $s$.

III. SCHEDULING ON A REFERENCE TRAJECTORY

A. Problem Statement

Consider the block diagram of Fig. 1. This figure shows a standard unity feedback configuration in which the command trajectory $r^\gamma$ is generated by passing a reference signal $u^\gamma$ through a model of the plant $P_m$. This reference control signal may be the outcome of a nonlinear optimal control problem or some other off-line design process. The control input $u$ to the actual plant $P$ then consists of the reference control $u^\gamma$ and a small perturbational control $\delta u$. In the ideal situation of no modeling errors, disturbances, or other uncertainties, one has that the perturbation control $\delta u = 0$, and perfect command tracking is achieved, i.e., $y = r^\gamma$.

Such perfect knowledge is rare, hence the need for feedback and compensator design. Now consider the block diagram of Fig. 2. This diagram represents the feedback system of Fig. 1 in the presence of three modeling errors: 1) $\Delta_y$, unmodeled sensor dynamics; 2) $\Delta_u$, unmodeled actuator dynamics; and 3) $\Delta_P$, an artificial uncertainty which corresponds to a performance specification (e.g., [4]).

A gain scheduled approach to control design for Fig. 2 would
be as follows. Let the plant model $P_m$ be given by

$$\dot{x}(t) = f(x(t)) + Bu(t), \quad x(0) = x_o \in \mathbb{R}^n,$$

$$y(t) = Cx(t) \quad (4)$$

where it is assumed that $f(\cdot) \in C^2$ and $B, C \in \mathbb{R}^m \times \mathbb{R}^n$. These equations are quite general since many systems may be put into the above form by selecting variables as outputs and augmenting dynamics at the plant input. Applying the reference command input $u^*$, one has that

$$\dot{x^*}(t) = f(x^*(t)) + Bu^*(t), \quad x^*(0) = x^*_o \in \mathbb{R}^n$$

$$r^*(t) = y^*(t) = Cx^*(t).$$

Now define

$$\delta x(t) = x(t) - x^*(t),$$

$$\delta y(t) = y(t) - y^*(t),$$

$$\delta u(t) = u(t) - u^*(t).$$

Then linearizing the model dynamics (4) about the trajectory $x^*(\cdot)$ yields

$$\delta x(t) = \delta f(x^*(t)) \delta x(t) + B \delta u(t)$$

$$+ (\Delta u) \delta x(t)$$

$$\delta x(0) = x_o - x^*_o,$$

$$\delta y(t) = C \delta x(t) \quad (6)$$

Then the parameter is time. Thus, the time-invariant design plant (7), (8) may be viewed as a frozen-parameter plant with parameter $\tau$. Now it would be impractical to require that a design be performed for all values of time over an infinite horizon without some additional simplifications. Thus, practical considerations would limit this approach to either finite horizon control problems or "asymptotically" periodic/constant reference trajectories.

Now suppose that the command trajectory $r^*$ is not known beforehand. Rather, it may be taken from a known family of possible command trajectories. This family, in turn, implies a family of possible values of the reference state-trajectory $x^*$. That is for some known set $X$, one has that $x^*(t) \in X$. In this case, the time-varying plant (5), (6) may be viewed as parameter-varying with $n$ parameters being the reference state trajectory. Thus, performing a design which guarantees frozen parameter stability for all admissible values of $x^*(t)$ would guarantee the desired robust stability and robust performance for all frozen-values of time.

Since the original plant is nonlinear and time-varying, none of the desired feedback properties— including nominal stability—of the frozen time designs may be present in the overall gain scheduled system. In the following section, conditions are given which guarantee the robust stability and robust performance of the global gain scheduled design.

B. Stability, Robustness, and Performance Analysis

Suppose that one has carried out the gain scheduled design procedure discussed in Section III-A. That is, at each instant of time, one has designed a finite-dimensional compensator which stabilizes the feedback configuration of Fig. 3. This leads to a collection of linear time-invariant compensators indexed by the time-parameter $\tau$. However, upon implementation, this collection of frozen-time compensators becomes a single time-varying compensator. Let this resulting time-varying compensator have the following state-space realization:

$$\dot{z}(t) = A(t)z(t) + B(t)\xi(t),$$

$$\delta u(t) = C(t)\xi(t).$$

Substituting the plant equations (5), (6) along with the above compensator equations, one has that the feedback equations of Fig. 2 take the form

$$z(t) = \bar{A}(t)z(t) + \int_0^t B(t)\Delta(t - \tau)C(t)\xi(t)\, d\tau$$

$$+ \delta F(t, z(t)) + d(t) \quad (9)$$

where

$$\bar{A}(t) \triangleq A(t) + B(t)C(t)B(t).$$

Let $\Delta(t)$ denote the nonlinear time-varying perturbational plant (5), (6), and let $\Delta_P$ denote the linear frozen-time plant

$$\Delta_P \triangleq \Delta(t) \triangleq \left( \begin{array}{c} \Delta_u(t) \\ 0 \end{array} \right)$$

$$\Delta(t) \triangleq \left( \begin{array}{c} \Delta_u(t) \\ 0 \end{array} \right)$$

$$\delta F(t, z(t)) \triangleq \left( \begin{array}{c} \delta f(t, x(t), x^*(t)) \\ 0 \end{array} \right).$$
$$d(t) \triangleq \begin{pmatrix} B(\Delta \mu^*)(t) \\ -B(t)(I - \Delta \zeta)^{-1} \Delta \mu^*(t) \end{pmatrix}.$$  

In the context of scheduling on a reference trajectory, stability of (9) means that the overall gain scheduled feedback system maintains the desired feedback properties. The stability of (9) will be shown as follows. Recall that the compensator was designed so that the VIDE

$$\dot{\xi}(t) = \tilde{A}(t)\xi(t) + \int_0^t \tilde{B}(t)\Delta(t - \tau)\xi(\tau) d\tau$$  \hspace{1cm} (10)

is stable for all frozen $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$. Using results from Section II, it is shown that (10) is exponentially stable for sufficiently slow time-variabilities. Given this time-varying exponential stability, a Lyapunov functional for (10) is constructed. This generalizes the concept of "convex transients of Lyapunov" for ordinary differential equations (e.g., [1], [5]). This Lyapunov functional is then used to give guaranteed stability margins for (9).

**Step 1—Slowly Time-Varying Stability of (10):** Since (10) is precisely the class of equations addressed in Section II, one can use Theorem 2.2 to guarantee stability for sufficiently slow time-variations as follows.

**Assumption 3.1:** The matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are bounded and globally Lipschitz continuous.

**Assumption 3.2:** There exists a constant $\beta > 0$ such that

$$s \rightarrow (s - \tilde{A}(\tau) - \tilde{B}(\tau) \dot{\Delta}(\tau) \tilde{C}(\tau))^{-1} \in \tilde{\mathbb{D}}^{\beta q}(-\beta), \quad \forall \tau \in \mathbb{R}^+,$$

$$\dot{\Delta} \in \tilde{\mathbb{D}}^{m \times \beta}(-\beta).$$

The following theorem is a direct consequence of Theorem 2.2.

**Theorem 3.1:** Consider the linear time-varying VIDE (10) under Assumptions 3.1 and 3.2. Under these conditions, (10) is exponentially stable for sufficiently slow time-variations in $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ (i.e., for a sufficiently small value of $k$ in (3)).

In terms of the reference state-trajectory $x^*$, this slowness condition on the dynamics of (10) states that $x^*$ itself should vary slowly. In light of the discussion in Section III-A, this comes as no surprise since the designs were based on frozen values of $x^*(t)$.

**Step 2—Construction of Lyapunov Functional:** Assume now that one has satisfied Theorem 3.1 to guarantee the time-varying stability of (10). Let $s(t; \phi, t_0)$ denote the solution to (10) with initial conditions $(\phi, t_0)$. From the definition of exponential stability, there exist constants $m$, $\lambda$, $\beta > 0$, where $\beta \geq \lambda$, such that for any initial condition $(\phi, t_0)$

$$|s(t; \phi, t_0)| \leq m e^{-\lambda(t - t_0)} \|W_1(\cdot, \phi)\|_{\mathbb{R}}.$$  \hspace{1cm} (11)

**Theorem 3.2:** Consider the linear time-varying VIDE (10). Suppose that (10) is exponentially stable as in (11). Under these conditions given any $\gamma \in (0, 1)$, there exists a function $V: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ which satisfies

$$\|W_1(\cdot, \phi)\|_{\mathbb{R}} \leq V(x, t) \leq m \|W_1, \phi\|_{\mathbb{R}}.$$  \hspace{1cm} (12)

$$|V(x, t) - V(x', t)| \leq m \|W_1(\cdot, \phi)(x - x')\|_{\mathbb{R}}.$$  \hspace{1cm} (13)

Furthermore, let $\dot{V}(t)$ denote $V$ evaluated along trajectories of (10), i.e.,

$$\dot{V}(t) \triangleq \frac{d}{dt} V(x(t; \phi, t_0), t), \quad t \geq t_0.$$  \hspace{1cm} (14)

Then $V$ satisfies

$$\mathcal{D}^\gamma \dot{V}(t) \leq -\gamma \lambda \|W_1(\cdot, \phi)\|_{\mathbb{R}}, \quad t \geq t_0.$$  \hspace{1cm} (15)

**Proof:** It is first shown that the exponential stability condition (11) also implies that

$$\|s(t; \phi, t_0)\|_{\mathbb{R}} \leq m e^{-\lambda(t - t_0)} \|W_1(\cdot, \phi)\|_{\mathbb{R}}.$$  \hspace{1cm} (16)

Towards this end, recall that

$$\|s(t; \phi, t_0)\|_{\mathbb{R}} \leq \max \left( \sup_{\xi \in \mathbb{R}^+} e^{-\beta(t - t_0)} \|s(\xi; \phi, t_0)\|_{\mathbb{R}} \right).$$

Thus

$$\|s(t; \phi, t_0)\|_{\mathbb{R}} \leq \max \left( \sup_{\xi \in \mathbb{R}^+} e^{-\beta(t - t_0)} \|s(\xi; \phi, t_0)\|_{\mathbb{R}} \right).$$

The remainder of the proof follows closely those of "convex transients of Lyapunov" (e.g., [1], [5]) for nonlinear ordinary differential equations. Let $\gamma \in (0, 1)$. Then define

$$V(x, t) \triangleq \sup_{r \geq t} (e^{\lambda(r - t)} \|W_1, s(\cdot, \phi)\|_{\mathbb{R}}).$$

The bounds (12) on $V$ follow immediately from the exponential stability condition (11). To show that $V$ satisfies the Lipschitz condition (13), note that

$$V(x, t) - V(x', t) = \sup_{r \geq t} (e^{\lambda(r - t)} \|W_1, s(\cdot, \phi)(x, t)\|_{\mathbb{R}} - s(\cdot, \phi)(x, t)).$$

Repeating this argument with $x$ and $x'$ reversed, it follows that:

$$V(x, t) - V(x', t) \leq \sup_{r \geq t} (e^{\lambda(r - t)} \|W_1, s(\cdot, \phi)(x, t) - s(\cdot, \phi)(x', t)\|_{\mathbb{R}}).$$

By linearity of (10),

$$s(\cdot, \phi)(x, t) - s(\cdot, \phi)(x', t) = s(\cdot, \phi)(x - x', t).$$

Thus

$$V(x, t) - V(x', t) \leq \sup_{r \geq t} (e^{\lambda(r - t)} \|W_1, s(\cdot, \phi)(x - x', t)\|_{\mathbb{R}}).$$

Condition (13) then follows from exponential stability.
Finally, to prove the negative definiteness condition (14), recall that
\begin{align*}
\mathcal{D}^+ \hat{V}_{10}(t) &\triangleq \limsup_{h \to 0^+} \frac{V(s(\cdot; \phi, t_0), t + h) - V(s(\cdot; \phi, t_0), t)}{h}.
\end{align*}
Evaluating this equation term by term gives
\begin{align*}
V(s(\cdot; \phi, t_0), t + h) &= \sup_{r \geq t + h} (e^{\lambda r(t - t + h)}) \|\mathcal{W}_r \mathbf{s}(\cdot; \cdot; \cdot; \cdot; \phi, t_0, t + h)\|_{\mathbb{A}}) \|_{\mathbb{A}}.
\end{align*}
and
\begin{align*}
V(s(\cdot; \phi, t_0), t) &= \sup_{r \geq t} (e^{\lambda r(t - t)}) \|\mathcal{W}_r \mathbf{s}(\cdot; \cdot; \cdot; \cdot; \phi, t_0, t)\|_{\mathbb{A}}.
\end{align*}
Exploiting the properties of $V$ and $\mathbf{s}$,
\begin{align*}
V(s(\cdot; \phi, t_0), t) &\geq \sup_{r \geq t} (e^{\lambda r(t - t)}) \|\mathcal{W}_r \mathbf{s}(\cdot; \cdot; \cdot; \cdot; \phi, t_0, t)\|_{\mathbb{A}}.
\end{align*}
Thus,
\begin{align*}
V(s(\cdot; \phi, t_0), t + h) &\leq e^{-\lambda h} V(s(\cdot; \phi, t_0), t).
\end{align*}
Condition (14) follows from using the above equation into the definition of $\mathcal{D}^+ \hat{V}_{10}(t)$ and letting $h \to 0^+$ (cf. [5]).

As mentioned earlier, Theorem 3.2 represents a type of “converse theorem of Lyapunov.” It is noted that the existence of a function which satisfies (12)-(14) can be used to prove uniform exponential stability of (10). Thus, Theorem 3.2 is also a statement of the equivalence of exponential stability and existence of Lyapunov functions (as in [1], [5]). Finally, it is noted that Theorem 3.2 does not require that the exponential stability of (10) is due to slow time-variations.

**Step 3—Proof of Stability of the Overall Gain Scheduled**

\begin{align*}
\mathcal{D}^+ \hat{V}_{10}(t) &= \limsup_{h \to 0^+} \frac{V\left(s(\cdot; \phi, t_0), t + h\right) - V\left(s(\cdot; \phi, t_0), t\right)}{h} \\
&= \limsup_{h \to 0^+} \frac{V\left(s(\cdot; s(\cdot; \phi, t_0), t), t + h\right) - V\left(s(\cdot; \phi, t_0), t\right)}{h} \\
&\quad + \limsup_{h \to 0^+} \frac{V\left(s(\cdot; s(\cdot; \cdot; \phi, t_0), t), t + h\right) - V\left(s(\cdot; s(\cdot; \cdot; \phi, t_0), t), t\right)}{h} \\
&\leq -\gamma \lambda \|\mathcal{W}_t \mathbf{s}(\cdot; \cdot; \cdot; \cdot; \phi, t_0)\|_{\mathbb{A}} + \limsup_{h \to 0^+} \frac{m \|\mathcal{W}_t + h \mathbf{s}(\cdot; \cdot; \cdot; \cdot; \phi, t_0) - s(\cdot; s(\cdot; \cdot; \cdot; \phi, t_0), t)\|_{\mathbb{A}}}{h}.
\end{align*}

**System:** Recall that the feedback configuration of Fig. 2 leads to dynamics of the form in (9)
\begin{align*}
\dot{z}(t) &= \mathbf{A}(t) \mathbf{z}(t) + \int_0^t \mathbf{B}(t) \Delta(t - r) \mathbf{C}(r) \mathbf{d}r \\
&\quad + \delta F(t, z(t)) + d(t).
\end{align*}
In light of Steps 1 and 2, this equation may be viewed as perturbations ($\delta F$ and $d$) on an exponentially stable time-varying VIDE. Using the Lyapunov functional of Theorem 3.2, conditions will be placed on $\delta F$ and $d$ to guarantee the boundedness of solutions to (9).

First, the following assumption is made on $\delta F$.

**Assumption 3.3:** There exists a constant $k_{\delta F} \geq 0$ such that
\begin{align*}
|\delta F(t, z)| &\leq k_{\delta F} |z|^2, \\
&\quad \forall t \in \mathbb{R}^+, \forall z \in \mathbb{R}^q.
\end{align*}

This quadratic bound reflects that $\delta F$ is a residual from a linearization. Regarding the global nature of Assumption 3.3, it is noted that such a bound will exist in case all of the second derivatives of $f(\cdot)$ are globally bounded (as in sinusoids, quadratic terms, etc.). In case the second derivatives are only locally bounded, then Assumption 3.3 will be valid over a bounded—yet arbitrarily large, depending on $k_{\delta F}$—region. In this case, it is straightforward to modify appropriately the subsequent proofs. Since the neighborhood of local validity of Assumption 3.3 may be made arbitrarily large (depending on the selection of $k_{\delta F}$), it may be made sufficiently large to include the “practical operating range” of the nonlinear dynamics. Thus, it is felt that the global nature of Assumption 3.3 is not very restrictive.

The stability of (9) is now addressed. Let $s'(\cdot; \phi, t_0)$ denote the solution to (9) with initial condition $(\phi, t_0)$.

**Theorem 3.3:** Consider the nonlinear VIDE (9) with initial condition $(\phi, t_0)$ under Assumptions 3.1–3.3. Let the linear time-varying VIDE (10) be exponentially stable as in (11). Let $V$ be defined as in Theorem 3.2. Then given any $\gamma' \in (0, 1)$
\begin{align*}
\|\mathcal{W}_t \mathbf{s}(\cdot; \cdot; \cdot; \cdot; \phi, t_0)\|_{\mathbb{A}} &\leq \frac{\gamma \lambda}{m k_{\delta F}} \gamma',
\end{align*}
and
\begin{align*}
\|d\|_{L_\infty} &\leq \frac{(\gamma \lambda)^2}{m k_{\delta F}^2} \left(1 - \gamma\right) \gamma',
\end{align*}
together imply that
\begin{align*}
\hat{y}(t; \phi, t_0) &\leq \frac{2 \gamma \lambda}{m k_{\delta F}} \gamma', \\
t \geq t_0.
\end{align*}
**Proof:** Let $V$ be defined as in Theorem 3.2, and let $\hat{V}(t)$ denote $V$ evaluated along trajectories of (9), i.e.,
\begin{align*}
\hat{V}(y)(t) &\triangleq V(s'(\cdot; \phi, t_0), t), \\
t \geq t_0.
\end{align*}
It is first shown that for $t \geq t_0$,
\begin{align*}
\mathcal{D}^+ \hat{V}(y)(t) &\leq -\gamma \lambda \hat{V}(y)(t) - m k_{\delta F} \hat{V}(y)(t) + m \|d\|_{L_\infty}.
\end{align*}
By definition of $\mathcal{D}^+ \hat{V}(y)(t)$,
\begin{align*}
\mathcal{D}^+ \hat{V}(y)(t) &\leq -\gamma \lambda \hat{V}(y)(t) + m k_{\delta F} \hat{V}(y)(t) + m \|d\|_{L_\infty}.
\end{align*}
Applying standard bounding techniques in conjunction with Assumptions 3.1, 3.2, it can be shown that
\begin{align*}
\hat{y}(t + h; s'(\cdot; \phi, t_0), t) - s(t + h; s'(\cdot; \phi, t_0), t) &\leq h (k_{\delta F}) \hat{y}(t; s'(\cdot; \phi, t_0), t)^2 + \|d\|_{L_\infty} + \text{higher order terms in } h,
\end{align*}
which, when substituted into the above bound on $\mathbb{D}^t \dot{V}_{\phi}(t)$, leads to the desired bound (18). Now given $\gamma \in (0, 1)$, let $d$ be bounded as in (16). Furthermore, suppose that at time $t$,

$$
\dot{V}_{\phi}(t) = \frac{\gamma \lambda}{mk_{SF}} \gamma'.
$$

Substituting this value of $\dot{V}_{\phi}(t)$ and the bound (16) on $d$ into (18), it follows that:

$$
\mathbb{D}^t \dot{V}_{\phi}(t) \leq 0.
$$

Thus, by definition of the Dini derivative, if $d$ is bounded as in (16) and

$$
\dot{V}_{\phi}(t_0) \leq \frac{\gamma \lambda}{mk_{SF}} \gamma'
$$

then

$$
\dot{V}_{\phi}(t) \leq \frac{\gamma \lambda}{mk_{SF}} \gamma'
$$

holds for all time. These conditions immediately translate into (15)–(17) which completes the proof.

Theorem 3.3 can be interpreted as a type of small-signal finite-gain stability result [1], [11]. It states that provided the disturbance $d$ is sufficiently small, the mapping $d \mapsto s(\cdot; \phi, t_0)$ is finite gain stable with gain $\frac{\gamma \lambda}{\gamma' - 1}$.

Thus, the definition of $d$:

$$
d(t) \triangleq \begin{cases} 
B(\Delta u^*)(t) \\
-Br_f(t)(I - \Delta p)^{-1} \Delta x^*(t)
\end{cases}
$$

Thus, it can be seen that the condition "$d$ sufficiently small" essentially states the intuitive condition that the reference trajectory $x^*$ and $r^*$ should not excite the unmodeled actuator or sensor dynamics.

To summarize, it has been shown that a gain scheduled approach applied to the feedback system of Fig. 2 has guaranteed robustness and performance properties under the following conditions. First, it is required that the reference trajectory $x^*$ is sufficiently slow. This comes as no surprise since the gain scheduled designs are based on linear time-invariant approximations to the plant. The restriction of slow variations simply states that such a frozen-time approximation should be accurate. Since the system is actually nonlinear, the internal stability is only local.

As the nonlinearities approach zero (i.e., $k_{SF} \to 0$), one has that the internal stability approaches global internal stability. Again, the restriction that nonlinearities impose are reminiscent that the design plants are linear time-invariant. The nonlinearities place another restriction on the feedback system, this time on the reference trajectories $x^*$ and $r^*$. Namely, it is required that these reference trajectories do not excite the unmodeled dynamics. For example, if the reference control trajectory $u^*$ has significant frequency content in the region of unmodeled actuator dynamics, then one cannot make demands on the resulting stability and performance of the closed-loop gain-scheduled system. In fact, since the reference control trajectory is fed forward to the plant, it is unlikely that any control strategy can remedy this situation.

**IV. SCHEDULING ON THE PLANT OUTPUT**

**A. Problem Statement**

Consider the plant model given by

$$
\begin{align*}
\frac{d}{dt} \begin{bmatrix} y(t) \\
z(t) \end{bmatrix} &= f(y(t), z(t)) + Bu(t), \\
y(t) &\in \mathbb{R}^n, \quad z(t) &\in \mathbb{R}^{n-m}, \quad u(t) &\in \mathbb{R}^m
\end{align*}
$$

where the plant output $y$ is a state variable. The following assumptions are made on (19).

**Assumption 4.1:** $f: \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ is at least twice continuously differentiable over all of $\mathbb{R}^m \times \mathbb{R}^{n-m}$ and satisfies

$$
f(0, 0) = 0.
$$

**Assumption 4.2:** There exist unique continuously differentiable functions $u_{eq}$ and $z_{eq}$ such that

$$
\begin{align*}
u_{eq} &: \mathbb{R}^m \to \mathbb{R}^{m}, \\
z_{eq} &: \mathbb{R}^m \to \mathbb{R}^{n-m},
\end{align*}
$$

and

$$
0 = f(y, z_{eq}(y)) + Bu_{eq}(y).
$$

Assumption 4.2 states that there exists a family of equilibrium conditions parameterized by the output $y$. In terms of gain scheduling, each of these equilibrium conditions is a possible "operating condition."

A gain scheduled approach to controlling (19) would be as follows. The plant linearized about a possible operating point $y_o$ is given by (suppressing the explicit dependence on time, $t$)

$$
\begin{align*}
\frac{d}{dt} \begin{bmatrix} y - y_o \\
z - z_{eq}(y_o) \end{bmatrix} &= \mathbb{D}f(y_o, z_{eq}(y_o)) \begin{bmatrix} y - y_o \\
z - z_{eq}(y_o) \end{bmatrix} + B(u - u_{eq}(y_o)) \quad (20)
\end{align*}
$$

Thus, at each operating point, one would design a compensator based on a local linear time-invariant approximation (20). This procedure would result in a family of linear time-invariant compensators $(A_k(y_o), B_k(y_o), C_k(y_o))$ parameterized by the operating condition $y_o$. These compensators are then used as in Fig. 4. In this figure, the current operating condition is instantaneously updated as the current plant output. Thus, the compensator dynamics would evolve as

$$
\begin{align*}
\dot{x_k}(t) &= A_k(y(t))x_k(t) + B_k(y(t))u(t), \\
\dot{u}(t) &= C_k(y(t))x_k(t).
\end{align*}
$$

Now this collection of gain scheduled designs has the property that for each linearized frozen operating condition, the feedback system has desirable stability, robustness, and performance properties. However, the actual gain scheduled system will have a time-varying (endogenous) scheduling variable evolving under nonlinear dynamics. In the following section, conditions are given for the nominal stability, robust stability, and robust performance properties of the fixed operating point designs to carry over to the actual gain scheduled system.

**B. Stability, Robustness, and Performance Analysis**

Consider now the feedback configuration of Fig. 5. This figure represents the nonlinear gain scheduled system of Fig. 4 in the presence of plant input unmodeled dynamics and artificial unmodeled dynamics which represent performance specifications. In this section, it is shown how one can guarantee that the robust
stability of the frozen operating condition designs carries over to the full gain scheduled system of Fig. 5.

Following the gain scheduling design procedure discussed in Section IV-A, the feedback equations of Fig. 5 take the form

\[
\dot{x}(t) = \bar{A}(y(t))x(t) + \int_0^t \bar{B}(y(t))\Delta(t-\tau)\bar{C}(y(\tau))x(\tau)\,d\tau + \delta F(x(t)) + (gx)(t) + d(t) \tag{21}
\]

where

\[
x(t) = \begin{pmatrix} y(t) \\ z(t) - z_{eq}(y(t)) \\ x_\delta(t) \end{pmatrix},
\]

\[
\bar{A}(y) = \begin{pmatrix} 0 & \mathcal{D}f_y(y, z_{eq}(y)) \\ \mathcal{D}f_y(y, z_{eq}(y)) - \mathcal{D}f_z(y, z_{eq}(y)) & \mathcal{D}f_y(y, z_{eq}(y)) - \mathcal{D}f_z(y, z_{eq}(y)) & \mathcal{D}f_z(y, z_{eq}(y)) \\ -B_\delta(y) & 0 & 0 \end{pmatrix},
\]

\[
\bar{B}(y) = \begin{pmatrix} B_y \\ B_\delta(y) \\ 0 \end{pmatrix},
\]

\[
\Delta(t) = \begin{pmatrix} \Delta_u(t) \\ \Delta_x(t) \end{pmatrix},
\]

\[
d(t) = \int_0^t \begin{pmatrix} 0 \\ 0 \\ B_\delta(y(t)) \end{pmatrix} (I - \Delta_p)^{-1}(t) (I - \Delta_p)^{-1}(t) \,d\tau,
\]

\[
(gx)(t) = \begin{pmatrix} B_y \\ B_\delta(y(t)) \end{pmatrix} \int_0^t \Delta_u(t-\tau) \Delta_x(\tau) \,d\tau,
\]

and

\[
\delta F(x(t)) = \begin{pmatrix} \delta f_y(y, z) \\ \mathcal{D}f_z(y, z) - \mathcal{D}f_z(y, z_{eq}(y)) - \mathcal{D}f_z(y, z) \end{pmatrix}.
\]

with

\[
\delta f(y, z) = f(y, z_{eq}(y)) + \mathcal{D}f(y, z_{eq}(y))(z - z_{eq}(y)).
\]

Throughout the above equations, the subscripts \(y\) and \(z\) denote decomposition of the matrix functions into their \(y\) and \(z\) components, respectively.

One important remark regarding the form of (21) is as follows. The closed-loop "dynamics matrix" \(\bar{A}(y)\) differs from the closed-loop dynamics matrix which would occur from applying the compensator dynamics \(\bar{A}_c(y), \bar{B}_c(y), \bar{C}_c(y), \bar{C}_c(y)\) to the linearized plant dynamics (20). This difference (discussed further in [6]) leads to the question of what one should use as "design matrices" for the plant.

In the remainder of this section, conditions are given which guarantee the stability of (21). In terms of gain scheduling, this means that the robustness and performance properties of the frozen operating point designs carry over to the full gain-scheduled system of Fig. 5. The stability of (21) will be shown using techniques similar to those of Section III-A. More precisely, using results from Section II-B, a Lyapunov functional is constructed for the quasi-linear VIDE

\[
\dot{x}(t) = \bar{A}(y(t))x(t) + \int_0^t \bar{B}(y(t))\Delta(t-\tau)\bar{C}(y(\tau))x(\tau)\,d\tau.
\]

Recall that the compensator was designed so that (22) is stable for all frozen \(\bar{A}, \bar{B}, \bar{C}\). This Lyapunov functional is then used to give guaranteed stability margins for (21).

Step 1—Local Stability of (22): The following assumptions are made on (22).

Assumption 4.3: The matrices \(\bar{A}, \bar{B}, \bar{C}\) are bounded and globally Lipschitz continuous.

Assumption 4.4: There exists a constant \(\beta > 0\) such that

\[
\Delta \in \tilde{\mathcal{D}}^\beta(-2\beta), \quad \forall \tau \in \mathbb{R}^+.
\]

The following theorem is a direct consequence of Theorem 2.2.

Theorem 4.1: Consider the VIDE (22) with initial condition \((\phi, t_0)\) under Assumptions 4.3 and 4.4. Under these conditions, there exist constants \(m, \lambda, \epsilon > 0\) such that

\[
y \leq \epsilon, \quad \forall t \in [t_0, t_0 + T] \Rightarrow \|x(t)\| = \|\mathcal{D}f(x(t))\| \leq m e^{-\lambda(t-t_0)} \|W_{\epsilon, \delta}\|, \quad \forall t \in [t_0, t_0 + R].
\]

As is Theorem 3.1, the slowness condition of the dynamics of (22) is because the designs were based on frozen values of the output \(y\).

The local exponential stability of (22) is now addressed.

Lemma 4.1: Consider the VIDE (22) with initial condition \((\delta, t_0)\) under Assumptions 4.3 and 4.4. Under these conditions, there exists a continuous monotonically increasing function \(f: \mathbb{R}^+ \to \mathbb{R}^+\) such that \(f(0) = 1\) and

\[
\|W_{\epsilon, \delta}\| \leq f(t-t_0)\|W_{\epsilon, \delta}\|, \quad \forall t \in [t_0, t_0 + R].
\]

Proof: The solution to (22) is given by

\[
x(t) = x_0 + \int_{t_0}^t \bar{A}(y(\tau))x(\tau) + \int_{t_0}^{\tau} \bar{B}(y(\tau))\Delta(\tau-\xi) \,d\xi + \int_{t_0}^{\tau} F(\tau) \,d\tau.
\]

C(y(\xi))x(\xi) \,d\xi + \int_{t_0}^{\tau} F(\tau) \,d\tau.
where
\[
F(\tau) \overset{\text{def}}{=} \int_{0}^{\tau} B(\gamma(\tau)) \Delta(\tau - \xi) \varphi(\xi) \, d\xi \, d\tau.
\]

Switching the order of integration yields
\[
x(t) = x_0 + \int_{t_0}^{t} \left( A(\gamma(\tau)) + \int_{0}^{\tau} B(\gamma(\xi)) \Delta(\xi - \tau) \varphi(\tau) \right) \, d\tau + \int_{t_0}^{t} F(\tau) \, d\tau.
\]

Now in the proof of Theorem 2.2 in [7], [8], it was shown that \( F \) can be bounded by a decaying exponential
\[
\|F(\tau)\| \leq \alpha e^{-\beta(\tau - t_0)} \|W_{t_0}, \varphi\|_a.
\]
Furthermore, Assumptions 4.3 and 4.4 guarantee that for some \( K \geq 0, \)
\[
\|A(\gamma(\tau)) + \int_{0}^{\tau} B(\gamma(\xi)) \Delta(\xi - \tau) \varphi(\tau)\| \leq K,
\]
\( \forall \tau \geq t_0. \)

Using these bounds and applying the Bellman–Gronwall inequality (e.g., [3]) to bound \( \|\gamma(\tau)\| \), it follows that
\[
\|\gamma(\tau)\| \leq f(t - t_0) \|W_{t_0}, \varphi\|_a, \quad t \geq t_0,
\]
where \( f \) is a continuous monotonically increasing function such that \( f : \mathbb{R}^+ \to \mathbb{R}^+ \). Finally, Lemma 4.1 follows using the same arguments found in the proof of Theorem 3.2.

**Theorem 4.2:** Consider the VIDE \((\phi, t_0)\) under Assumptions 4.3 and 4.4. Let \( m, \lambda, \) and \( \epsilon \) be as in Theorem 4.1, let \( f \) be as in Lemma 4.1, and let \( L_\gamma \) be such that
\[
\|\gamma(t)\| \leq L_\gamma \|W_{t_0}, \varphi\|_a, \quad t \geq t_0.
\]

Under these conditions, given any \( \gamma \in (0, 1), \)
\[
\|W_{t_0}, \varphi\|_a \leq \rho \overset{\text{def}}{=} \frac{\gamma e}{L_\gamma \left( \frac{\ln m}{\lambda} + \delta \right)}, \quad (24)
\]
implies
\[
\|\gamma(t)\| \leq me^{-\lambda(\tau - t_0)} \|W_{t_0}, \varphi\|_a, \quad (25)
\]

**Proof:** Let \( \delta \) be such that
\[
\frac{f \left( \frac{\ln m}{\lambda} + \delta \right)}{f \left( \frac{\ln m}{\lambda} \right)} = \gamma.
\]

Then combining (23) with (24), it follows that
\[
\|\gamma(t)\| \leq \epsilon, \quad \forall t \in \left[ t_0, t_0 + \frac{\ln m}{\lambda} + \delta \right].
\]


during this interval, Theorem 4.1 assures that the state decays exponentially. However, the restriction (24) on the initial condition guarantees that at the end of the interval, say at time \( T, \)
\[
\|W_{t_0}, \varphi\|_a \leq \|W_{t_0}, \varphi\|_a.
\]

Repeating this process, it follows that \( \|\gamma(t)\| \leq \epsilon \) for all time which in turn implies (25).

In words, Theorem 4.2 states that if the quasi-linear dynamics \((22)\) are stable for all frozen \( y \), which is the case for gain scheduling, then \((22)\) is locally stable. It is stressed that the local nature of the stability is not due to the linearization. Rather, it is because of the time-derivations in the scheduling variable \( y \).

Thus, the local exponential stability approaches global stability as the relative rate of time-derivations decreases, \( L_\gamma \to 0 \), or the size of allowable time-derivations increases, \( \epsilon \to \infty \).

**Step 2—Construction of a Lyapunov Functional:** Let \( s(t; \Phi, t_0) \) denote the solution to \((22)\) with initial condition \( (\phi, t_0) \).

**Assumption 4.5:** Let \( m \) and \( \rho \) be as in Theorem 4.2. There exists a constant \( L \) such that
\[
\|W_{t_0}, \varphi\|_a \|W_{t_0}, \varphi\|_a \leq m \rho, \quad x, x' \in \mathbb{G}
\]
implies
\[
\left( A(\gamma(t)) \varphi(t) + \int_{0}^{t} B(\gamma(t)) \Delta(t - \tau) \varphi(t) \, d\tau \right) \]
\[
- \left( A(\gamma'(t)) \varphi(t) + \int_{0}^{t} B(\gamma'(t)) \Delta(t - \tau) \varphi(t) \, d\tau \right) \leq \frac{L}{2} \|W_{t_0}, \varphi(x - x')\|_a.
\]

**Theorem 4.3:** Consider the VIDE \((22)\) with initial condition \( (\phi, t_0) \) under Assumptions 4.3–4.5. Let \( m, \lambda, \) and \( \rho \) be as in Theorem 4.2. Under these conditions, given any \( \gamma \in (0, 1), \) there exists a function \( V : \mathbb{G} \times \mathbb{R}^n \to \mathbb{R}^+ \) which satisfies
\[
\|W_{t_0}, \varphi\|_a \leq V(x, t) \leq m \|W_{t_0}, \varphi\|_a,
\]
\[\forall \|W_{t_0}, \varphi\|_a \leq \rho, \quad (26)\]

\[
|V(x, t) - V(x', t)| \leq \frac{LV}{2} \|W_{t_0}, \varphi(x - x')\|_a,
\]
\[\forall \|W_{t_0}, \varphi\|_a, \|W_{t_0}, \varphi\|_a \leq \rho \quad (27)\]

where
\[
L_V = e^{(\lambda + L_\gamma)T}, \quad T = \frac{\ln m}{(1 - \gamma)\lambda}.
\]

Furthermore, let \( \tilde{V}(22)(t) \) denote \( V \) evaluated along trajectories of \((22)\), i.e.,
\[
\tilde{V}(22)(t) \overset{\text{def}}{=} V(s(\cdot; \Phi, t_0), t), \quad t \geq t_0.
\]

Then \( V \) satisfies
\[
D^+ \tilde{V}(22)(t) \leq -\gamma \lambda \|W_{t_0}, \varphi(\cdot; \Phi, t_0)\|_a, \quad t \geq t_0.
\]

**Proof:** Define
\[
V(x, t) \overset{\text{def}}{=} \sup_{t \geq t} e^{(\lambda + L_\gamma)T} \|W_{t_0}, \varphi(\cdot; \Phi, t_0, t)\|_a.
\]

The bounds (26) and (28) follow immediately from local exponential stability and methods similar to those used in the proof of Theorem 3.2. As for the Lipschitz condition of (27), methods similar to those used in the proof of Theorem 3.2 lead to
\[
|V(x, t) - V(x', t)| \leq \sup_{t \geq t} e^{(\lambda + L_\gamma)T} \|W_{t_0}, \varphi(\cdot; \Phi, t_0, t) - s(\cdot; \Phi, x', t)\|_a.
\]

Now exponential stability implies that the above supremum may be taken over the finite interval \( T \in [t, t + T] \) where \( T = \frac{\ln m}{(1 - \gamma)\lambda}. \)

Using the Bellman–Gronwall inequality with Assumption 4.5 to
bound the right-hand side of the previous equation then yields the desired result.

**Step 3—Proof of Stability of the Overall Gain Scheduled System:** Recall that the feedback configuration of Fig. 5 leads to dynamics of the form in (21)

\[ \dot{x}(t) = \bar{A}(y(t))x(t) + \int_0^t \bar{B}(y(t))\Delta(t - \tau) \]

\[ C(y(t))x(t) + \Delta F(x(t)) + (\bar{g}(x(t)) + d(t)). \]

This equation may be viewed as perturbations (\(\Delta F, \bar{g}, \) and \(d\)) on a locally exponentially stable quasi-linear VIDE. Using the Lyapunov functional of Theorem 4.3, conditions will be placed on \(\Delta F, \bar{g}, \) and \(d\) to guarantee the boundedness of solutions to (21).

**Assumption 4.6:** The linearization residual satisfies

\[ |\Delta F(x)| \leq k_{AF}|z - z_{eq}(y)|^2 (\leq |x|^2), \quad \forall x \in \mathbb{R}^p. \]

Note that the term \(\Delta F(x)\) represents a residual from a linearization of only the function \(z = f(y, z)\) of the original plant model (19). This implies if the dynamics of (19) are linear in \(z\), then \(\Delta F = 0\). Thus, the notion of “the scheduling variable capturing the plant’s nonlinearities” is precisely quantified. As for the global nature of Assumption 4.6, see the discussion following Assumption 3.3.

**Assumption 4.7:** There exists a constant \(k_{SF}\) such that

\[ \|\bar{g}(x)\|_1 \leq k_{SF}\|x\|_2, \quad t \geq 0, \quad \forall x \in \mathbb{R}^p. \]

Thus, it has been shown that the gain scheduled system of Fig. 5 has guaranteed robust stability and robust performance properties under the following conditions. First are the familiar, and now quantitative, heuristic guidelines:

1) slow variations in the scheduling variable (Theorem 4.1);
2) small nonlinearities not captured by the scheduling variable (i.e., small \(k_{SF}\)).

In addition to these restrictions, one requires that

\[ \eta \overset{\text{def}}{=} \gamma L_{\bar{g}}k_{SF} > 0. \]

In words, this means that the degree of exponential stability must be large enough to overcome the nonlinear functional perturbation \(\bar{g}(x)\) in (21). In terms of the original definition of \(\bar{g}(x)\),

\[ B_y \begin{pmatrix} \angle \bar{g}(y(t)) \end{pmatrix} \int_0^t \Delta_{se}(t - \tau)u_{eq}(y(t))d\tau \]

this perturbation is a product of the equilibrium control exciting the unmodeled actuator dynamics. This restriction due to the presence of \(\bar{g}(x)\) is typical of such forms of “precompensation.” For example, a gain scheduled design, such as in Fig. 4, relies on the accuracy of the linearization through \(u_{eq}\). Thus, it is implicitly assumed that unmodeled dynamics occur at the compressor output—which is the case in the frozen-\(y\) designs. In terms of Fig. 4, the loop properties broken at: 1) the compressor output; and 2) the plant input have different properties. That is, it is possible that nondestabilizing unmodeled dynamics at the compressor output destabilize the feedback system when placed at the plant input. Note that by augmenting integrators at the plant input, one eliminates the equilibrium control feedback of Fig. 4. See [6] for further discussion of this point.

Finally, another restriction imposed by Theorem 4.4 is the familiar small-signal condition

\[ \|d\|_\infty \leq \frac{\eta^2}{L_{\bar{g}}k_{SF}} (1 - \gamma) \gamma'. \]

Recall that \(d\) is defined as

\[ d(t) \overset{\text{def}}{=} \int_0^t \begin{pmatrix} 0 \\ 0 \\ (I - \Delta_\bar{p})^{-1}(t - \tau)\bar{y}(\tau) \end{pmatrix}d\tau. \]

where \(\Delta_\bar{p}\) is an artificial uncertainty which represents a performance specification. In terms of typical linear control designs, \(\Delta_\bar{p}\) is large where good command following is expected (e.g., [10]). Thus, the small-signal condition on \(d\) states that the reference command should not have significant frequency content in the regions where poor command following is expected.

Before closing this section, it is noted that this analysis did not address the possible presence of unmodeled dynamics at the plant output. This is because including such unmodeled dynamics would result in feedback equations which become overly algebraically cumbersome. However, it is easy to show that the gain scheduled feedback system of Fig. 4 does maintain stability in the presence of possible plant output disturbances, i.e., the slightest disturbance will not destabilize the system. This stability margin is important since such a disturbance alters both the equilibrium control and the value of the scheduling variable used in the compensator. As would be expected, the margins of stability are proportional to the degree of frozen-\(y\) stability and inversely proportional to the time-variations in \(y\) and the size of the nonoutput nonlinearities.

**V. Concluding Remarks**

This paper has presented a formal analysis of two types of nonlinear gain scheduled control systems: 1) scheduling on a ref-
ference trajectory; and 2) scheduling on the plant output. In both cases, conditions were given which guarantee that certain stability, robustness, and performance properties of the frozen operating condition designs carry over to the overall gain scheduled design.

The main results may be summarized as follows. In the case of scheduling on a reference trajectory, given that the feedback system (9) is stable for all frozen values of time, the robust stability and robust performance is maintained provided that: 1) the reference trajectory varies slowly, and 2) the reference trajectory does not excite unmodeled dynamics. In the case of scheduling on the plant output (21), conditions were given which essentially verify and formalize the two common gain scheduling guidelines of "scheduling on a slow variable" and "capturing the plant's non-linearities." That is, in the limiting cases where the rate of the output time-variations approaches zero and the non-output non-linearities approach zero, then the feedback properties of the gain scheduled designs approach those of the frozen-time designs.

Although the systems analyzed were all of particular unity-feedback configurations, the tools developed are applicable to general feedback system whose dynamics may be put into the quasi-linear Volterra integrodifferential equations (9) or (21). In fact, it is this need for a quasi-linear form of the closed-loop dynamics which restricted the analysis to scheduling on a state-variable which is explicitly a plant output [as in (19)].

In this setting of stability theory for VIDE's, the two main methods of analysis which were generalized from their ordinary differential equation counterparts were Lyapunov-stability/exponential-stability equivalence [1] and small-signal finite-gain stability [1, 11].

The main limitations of these results are as follows. First, verification of these theorems requires hard-to-obtain information of the unmodeled dynamics, as discussed in [6], [8]. Second, even if the sufficient conditions are verified, they are apt to be conservative, which is typical of Lyapunov analyses of nonlinear systems.

In spite of these limitations, they do add valuable insight beyond the original heuristic guidelines. For example in Section III, the danger of reference trajectories exciting unmodeled dynamics was revealed in a quantitative manner. Furthermore, the theorems are useful in that they help to identify various parameters which, in turn, improve the feedback properties of the gain scheduled design. For example, in the case of scheduling on the plant output, one can now say in a quantitative manner what variables one should try to use as scheduling variables. Thus, it is believed that the contribution of these sufficient conditions does not lie in their explicit evaluation. Rather, they should be used to gain new insights for design purposes. For example in [6], the additional insight obtained from these results led to a design methodology for a certain class of nonlinear systems which guarantees stability for arbitrarily fast output variations and guarantees certain robustness properties.

REFERENCES

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