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Performance limitations in sensitivity reduction for nonlinear plants *

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Abstract: This paper investigates performance limitations imposed by 'non-minimum phase' characteristics of a nonlinear time-varying plant. A performance criterion is defined which, in the linear case, is analogous to minimizing the sensitivity over a given frequency band. It is shown that if the nonlinear plant is 'non-minimum phase', then the frequency-weighted sensitivity cannot be made arbitrarily small while keeping the overall sensitivity bounded. The non-minimum phasedness of the plant is stated in terms of a deficiency in its range. These results extend the familiar 'push/pop' phenomenon in sensitivity optimization to a nonlinear time-varying setting.

Keywords: Non-minimum phase systems; nonlinear systems; sensitivity reduction; performance limitations; disturbance rejection.

1. Introduction

One measure of performance in linear time-invariant (LTI) feedback systems is the magnitude of a frequency-weighted sensitivity transfer function. More precisely, let P be an LTI plant with transfer function $p(s)$ and let K be a stabilizing compensator with transfer function $k(s)$. Then given a range of frequencies, Ω, a measure of performance is

$$\mu(P, K, \Omega) := \sup_{\omega \in \Omega} |(1 + p(j\omega)k(j\omega))^{-1}|.$$

For further discussion and motivation of such performance measures, see [6] and references contained therein.

Given this performance measure, an important question is then what properties of the plant limit the the achievable performance? This has been considered in [7,10,16]. For example it has been shown for plants with rational transfer functions that if the plant is minimum phase, then $\mu(P, K, \Omega)$ can be made arbitrarily small while keeping the overall sensitivity bounded [16]. In case the plant has open right-half-plane zeros, then making $\mu(P, K, \Omega)$ arbitrarily small comes at the cost of making the overall sensitivity arbitrarily large [7]. This has been called the 'push/pop' or 'waterbed' phenomenon. These results were later extended to plants with irrational transfer functions in [10]. For further results on achievable performance, see [2,8,11].

In this paper, we consider analogous results for general nonlinear time-varying plants and compensators. Using a disturbance rejection interpretation of the above performance measure, an analogous performance measure is defined as follows. Let $\mathscr{D}$ be a given bounded class of finite-energy disturbances. Then define

$$\mu(P, K, \Omega, \mathscr{D}) := \sup_{d \in \mathscr{D}} \|(I + PK)^{-1} d\|_{\Omega},$$

where $\|\cdot\|_{\Omega}$ denotes the signal-energy distributed over the frequency range Ω. We show that if the plant is 'non-minimum phase', then making $\mu(P, K, \Omega, \mathscr{D})$ arbitrarily small comes at the cost of making response to an admissible disturbance arbitrarily large. In this nonlinear setting, the non-minimum phase property is expressed as a deficiency in the range of the plant. The goal of minimizing such performance measures for nonlinear plants is considered in [1].

The remainder of this paper is organized as follows. We establish some preliminary notation and definitions in Section 2. In Section 3, we define precisely our performance objective and show how a non-minimum phase plant limits the achievable performance. Finally, we discuss our definition of non-minimum phase for nonlinear plants in Section 4.

2. Preliminary definitions

Let $\mathscr{L}^2(S)$ denote the standard Hilbert space of real-valued measurable square-integrable functions defined on either $S=\mathscr{R}$ or $S=\mathscr{R}^+$ with norm $\|\cdot\|_{\mathscr{L}_p(S)}$. The inner product in $\mathscr{L}^2(S)$ is denoted $\langle f, g\rangle_{\mathscr{L}^2(S)}$. For $f\in\mathscr{L}^2(S)$, $\hat{f}$ denotes the Fourier transform of f. Note that via Parseval's identity (e.g., [5])

$$\langle f, g\rangle_{\mathscr{L}^2(S)} := \int_S f(t)g(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(-\mathrm{j}\omega)\hat{g}(\mathrm{j}\omega)\,\mathrm{d}\omega.$$

For $\Omega\subset\mathscr{R}$ with non-zero measure and $f\in\mathscr{L}^2(S)$, $\|f\|_\Omega$ is defined as

$$\|f\|_\Omega := \left(\frac{1}{2\pi}\int_\Omega|\hat{f}(\mathrm{j}\omega)|^2\,\mathrm{d}\omega\right)^{1/2}.$$

Note that for $f\in\mathscr{L}^2(\mathscr{R}^+)$, if $\|f\|_\Omega=0$ then via the analyticity of $\hat{f}$ in the open right-half complex plane, $f=0$ [12, Theorem 17.18].

Let f be a real-valued function on $\mathscr{R}^+$. Then $\Pi_T f$, $T\in\mathscr{R}^+$, denotes the function defined by

$$(\Pi_T f)(t) := \begin{cases} f(t), & 0\leqslant t\leqslant T,\\ 0, & t>T.\end{cases}$$

Let $\mathscr{L}_e^2(\mathscr{R}^+)$ denote the set of locally $\mathscr{L}^2$ functions, i.e.,

$$\mathscr{L}_e^2(\mathscr{R}^+) := \{f:\mathscr{R}^+\to\mathscr{R}^+:\ \Pi_T f\in\mathscr{L}^2(\mathscr{R}^+),\ \forall T\in\mathscr{R}^+\}.$$

A sequence $\{f_n\}\subset\mathscr{L}^2(S)$ is said to *weakly converge* to $f_0\in\mathscr{L}^2(S)$ if for all $g\in\mathscr{L}^2(S)$,

$$\lim_n\langle f_n, g\rangle_{\mathscr{L}^2(S)} = \langle f_0, g\rangle_{\mathscr{L}^2(S)}.$$

A mapping $H:\mathscr{L}_e^2(\mathscr{R}^+)\to\mathscr{L}_e^2(\mathscr{R}^+)$ is called an *I/O operator* if it is unbiased and causal. That is, $H0=0$, and

$$\Pi_T Hf = \Pi_T H\Pi_T f,\quad \forall T\in\mathscr{R}^+, \forall f\in\mathscr{L}^2(\mathscr{R}^+),$$

respectively.

Let H be an I/O operator. The *domain of H*, denoted $D(H)$, and the *range of H*, denoted $R(H)$, are defined as follows:

$$D(H) := \{f\in\mathscr{L}^2(\mathscr{R}^+):\ Hf\in\mathscr{L}^2(\mathscr{R}^+)\},$$
$$R(H) := \{Hf:\ f\in D(H)\}.$$

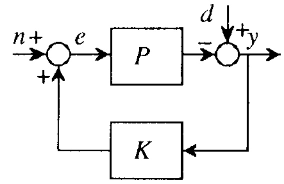

Fig. 1. Feedback system.

The $\mathscr{L}^2(\mathscr{R}^+)$*-closure* of $R(H)$ is denoted cl $R(H)$. The *weak closure* of $R(H)$ is denoted wk-cl $R(H)$. If $R(H)$ is convex (as in the linear case), then [3, Theorem V.1.4]

$$\mathrm{cl}\,R(H) = \text{wk-cl}\,R(H).$$

For further discussion on domains and ranges of nonlinear I/O operators, see [13,14].

Now consider the feedback system shown in Figure 1, where P and K are I/O operators. This system is said to be *well-posed* if for every $(d, n)\in\mathscr{L}_e^2(\mathscr{R}^+)\times\mathscr{L}_e^2(\mathscr{R}^+)$, there exist unique $(y, e)\in\mathscr{L}_e^2(\mathscr{R}^+)\times\mathscr{L}_e^2(\mathscr{R}^+)$ such that

$$y = d - Pe,$$
$$e = n + Ky,$$

and the mapping $(d, n)\mapsto(y, e)$ is causal [5,15]. The feedback system is called *stable* (also called S-stable in [4]) if

$$(d, n)\in\mathscr{L}^2(\mathscr{R}^+)\times\mathscr{L}^2(\mathscr{R}^+) \Rightarrow (y, n)\in\mathscr{L}^2(\mathscr{R}^+)\times\mathscr{L}^2(\mathscr{R}^+),$$

and there exists a continuous non-decreasing function $\psi:\mathscr{R}^+\to\mathscr{R}^+$ with $\psi(0)=0$ such that

$$\|y\|_{\mathscr{L}^2(\mathscr{R}^+)} + \|d\|_{\mathscr{L}^2(\mathscr{R}^+)} \leqslant \psi\left(\|d\|_{\mathscr{L}^2(\mathscr{R}^+)} + \|n\|_{\mathscr{L}^2(\mathscr{R}^+)}\right),\quad \forall(d, n)\in\mathscr{L}^2(\mathscr{R}^+)\times\mathscr{L}^2(\mathscr{R}^+).$$

In this case, the compensator K is said to *stabilize* the plant P.

3. Limitations in sensitivity reduction

We begin by defining a performance measure analogous to the maximum magnitude of the sensitivity transfer function over a specified frequency interval. Let the subset $\Omega\subset\mathscr{R}$ have nonzero measure and let $\mathscr{D}\subset\mathscr{L}^2(\mathscr{R}^+)$ be a bounded set of disturbances. Then for any plant, P, and stabilizing compensator, K, we define

$$\mu(P, K, \Omega, \mathscr{D}) := \sup_{d \in \mathscr{D}} \|(I + PK)^{-1} d\|_{\Omega}.$$

Informally, the performance measure $\mu(P, K, \Omega, \mathscr{D})$ simply expresses the maximum effect of a disturbance $d \in \mathscr{D}$ on the energy of

$$y = (I + PK)^{-1} d$$

in the frequency interval Ω.

This performance objective is particularly well-suited to nonlinear systems. It avoids induced norms since bounding the output norm by a linear function of the input norm can be too restrictive. Furthermore, it allows the class of disturbances to be defined as desired. For example, suppose $\mathscr{D}$ is defined by

$$\mathscr{D} = \left\{ d \in \mathscr{L}^2(\mathscr{R}^+) : \|d\|_{\mathscr{L}^2(\mathscr{R}^+)} \leqslant c_1 \text{ and } |d(t)| \leqslant c_2 \right\}.$$

Then the presence of the magnitude bound on $d(t)$ could be used to limit the disturbance to the 'operating region' of the nonlinear plant.

The main result is the following.

Theorem 3.1. *Let $\Omega \subset \mathscr{R}$ have non-zero measure and let $\mathscr{D} \subset \mathscr{L}^2(\mathscr{R}^+)$ be a bounded set of disturbances. Let $\{K_n\}$ be a sequence of I/O operators which stabilize the I/O operator P. Suppose that*

$$\mathscr{D} \not\subset \text{wk-cl } R(P).$$

Then $\mu(P, K_n, \Omega, \mathscr{D}) \to 0$ implies

$$\sup_n \sup_{d \in \mathscr{D}} \|(I + PK_n)^{-1} d\|_{\mathscr{L}^2(\mathscr{R}^+)} = \infty.$$

In case $\mathscr{D}$ is defined by

$$\mathscr{D} = \left\{ f \in \mathscr{L}^2(\mathscr{R}^+) : \|f\|_{\mathscr{L}^2(\mathscr{R}^+)} \leqslant 1 \right\},$$

and both P and K are LTI, it is easy to see that Theorem 3.1 degenerates to the results in [7,10].

Before proving Theorem 3.1, we establish a useful lemma.

Lemma 3.1. *Let $\Omega \subset \mathscr{R}$ have non-zero measure. Suppose the sequence $\{y_n\} \subset \mathscr{L}^2(\mathscr{R}^+)$ converges weakly to $y_0 \in \mathscr{L}^2(\mathscr{R}^+)$ and $\|y_n\|_{\Omega} \to 0$. Then $y_0 = 0$.*

Proof. For any $g \in \mathscr{L}^2(\mathscr{R}^+)$, let $\bar{g} \in \mathscr{L}^2(\mathscr{R})$ denote the 'backwards extension' of g. That is,

$$\bar{g}(t) = \begin{cases} g(t), & t \geqslant 0, \\ 0, & t < 0. \end{cases}$$

It is easy to see that $\{\bar{y}_n\} \subset \mathscr{L}^2(\mathscr{R})$ converges weakly to $\bar{y}_0 \in \mathscr{L}^2(\mathscr{R})$. Thus for any $f \in \mathscr{L}^2(\mathscr{R})$,

$$\langle \bar{y}_n, f \rangle_{\mathscr{L}^2(\mathscr{R})} \to \langle \bar{y}_0, f \rangle_{\mathscr{L}^2(\mathscr{R})}.$$

Pick $f \in \mathscr{L}^2(\mathscr{R})$ such that

$$\hat{f}(j\omega) = \begin{cases} \hat{\bar{y}}_0(j\omega), & \omega \in \Omega, \\ 0, & \text{otherwise}. \end{cases}$$

Then via Parseval's identity,

$$\langle \bar{y}_n, f \rangle_{\mathscr{L}^2(\mathscr{R})} \to \|y_0\|_{\Omega}^2.$$

However via Schwarz's inequality and Parseval's identity

$$\langle \bar{y}_n, f \rangle_{\mathscr{L}^2(\mathscr{R})} \leqslant \|y_n\|_{\Omega} \|f\|_{\Omega} \to 0.$$

It follows that $\|y_0\|_{\Omega} = 0$ which implies $y_0 = 0$. □

Proof of Theorem 3.1. We prove Theorem 3.1 by contradiction. Suppose that the sequence $\{K_n\}$ is such that

$$\sup_n \sup_{d \in \mathscr{D}} \|(I + PK_n)^{-1} d\|_{\mathscr{L}^2(\mathscr{R}^+)} \leqslant M < \infty.$$

Since $\mathscr{D} \not\subset \text{wk-cl } R(P)$, there exists a $d^* \in \mathscr{D}$ such that $d^* \notin \text{wk-cl } R(P)$. Define

$$y_n = (I + PK_n)^{-1} d^*.$$

Then d^* and y_n satisfy

$$y_n + PK_n y_n = d^*.$$

We first show that $PK_n y_n \in R(P)$. Since each K_n stabilizes P, we have

$$K_n y_n = K_n (I + PK_n)^{-1} d^* \in \mathscr{L}^2(\mathscr{R}^+).$$

Since

$$PK_n y_n = d^* - y_n \in \mathscr{L}^2(\mathscr{R}^+),$$

it follows that $PK_n y_n \in R(P)$.

Since the sequence $\{y_n\}$ is bounded, there exists a weakly convergent subsequence which we relabel $\{y_n\}$ [3, Theorem V.3.1]. Since $\| y_n \|_\Omega \to 0$, it follows from Lemma 3.1 that $\{y_n\}$ weakly converges to 0, and hence $d^* \in$ wk-cl $R(P)$ – a contradiction. □

4. Non-minimum phase nonlinear plants

The non-minimum phase condition on P stated in Theorem 3.1 is in terms of a deficiency in $R(P)$. More precisely, there exists a $d^* \in \mathscr{D}$ such that $d \notin$ wk-cl $R(P)$. This condition may be interpreted as an inability to construct a 'stable approximate inverse' of P. In the LTI case, this is analogous to the transfer function of P having a right-half-plane zero (e.g., [7]). For a differential geometric viewpoint of minimum phasedness in nonlinear plants, see [9].

Note that the given proof of Theorem 3.1 requires that the deficiency in $R(P)$ is in terms of the *weak closure of* $R(P)$ and not the norm closure. For a general nonlinear plant,

$$\text{cl}\ R(P) \subset \text{wk-cl}\ R(P),$$

with strict containment possible. As stated in Section 2, however, the two sets coincide when $R(P)$ is convex. The following proposition gives some additional insight into the structure of the set wk-cl $R(P)$.

Proposition 4.1. *Let P be a given I/O operator. Let F be any stable LTI I/O operator with a strictly proper rational transfer function. Let the sequence $\{y_n\} \subset R(P)$ converge weakly to $d \in \mathscr{L}^2$. Then for any $T \in \mathscr{R}^+$,*

$$\| \Pi_T F(d - y_n) \|_{\mathscr{L}^2(\mathscr{R}^+)} \to 0.$$

Proof (sketch). Represent d and y_n in terms of their Fourier series expansions:

$$d = \sum_k a_k \phi_k,$$

$$y_n = \sum_k b_{k,n} \phi_k,$$

where

$$\phi_k(t) := e^{jk(2\pi/T)t}.$$

Since the transfer function of F is strictly proper, it can be shown that

$$\lim_k \| \Pi_T F \phi_k \|_{\mathscr{L}^2(\mathscr{R}^+)} = 0.$$

Since any weakly convergent sequence is bounded, this implies that the quantity

$$\sup_n \left\| \Pi_T F \sum_{|k| \geq K} (a_k - b_{k,n}) \phi_n \right\|_{\mathscr{L}^2(\mathscr{R}^+)}$$

may be made arbitrarily small via appropriate choice of K. Since $\{y_n\}$ converges weakly to d, we have that for any k,

$$\lim_n b_{k,n} = a_k.$$

Using that

$$F(d - y_n) = F \sum_{|k| < K} (a_k - b_{k,n}) \phi_k + F \sum_{|k| \geq K} (a_k - b_{k,n}) \phi_k$$

then leads to the desired result. □

In words, Proposition 4.1 states that the closure and weak closure of $R(P)$ coincide modulo low-pass filtering and a finite time horizon.

References

[1] J.A. Ball and J.W. Helton, Nonlinear $\mathscr{H}^\infty$ control theory: A literature survey, *Proceedings of the International Symposium on the Mathematical Theory of Networks and Systems*, The Netherlands (1989).

[2] S. Boyd and C.A. Desoer, Subharmonic functions and performance bounds on linear time-invariant feedback systems, Memorandum No. UCB/ERL M85/51, Electronics Research Laboratory, University of California, Berkeley (June 1984).

[3] J.B. Conway, *A Course in Functional Analysis* (Springer-Verlag, New York, 1985).

[4] C.A. Desoer and C.A. Lin, Nonlinear unity feedback systems and Q-parameterization, in: A. Bensoussan and J.L. Lions, Eds., *Analysis and Optimization of Systems*, Lecture Notes in Control and Information Sciences No. 62 (Springer-Verlag, New York, 1984).

[5] C.A. Desoer and M. Vidyasagar, *Feedback Systems: Input–Output Properties* (Academic Press, New York, 1975).

[6] B.A. Francis, *A Course in $\mathscr{H}^\infty$-Optimal Control Theory* (Springer-Verlag, New York, 1987).

[7] B.A. Francis and G. Zames, On $\mathscr{H}^\infty$-optimal sensitivity theory for SISO feedback systems, *IEEE Trans. Automat. Control* **29** (1) (1984) 9–16.

[8] J.S. Freudenberg and D.P. Looze, Right half plane poles and zeros and design tradeoffs in feedback systems, *IEEE Trans. Automat. Control* **30** (6) (1985) 555–565.
[9] A. Isidori and C. Moog, On the nonlinear equivalent of the notion of transmission zeros, in: C.I. Byrnes and A. Kurzhanski, Eds., *Modelling and Adaptive Control*, Lecture Notes in Control and Information Sciences No. 105 (Springer-Verlag, New York, 1988).
[10] Y. Ohta, G. Tadmor, and S.K. Mitter, Sensitivity reduction over a frequency band, Report No. LIDS-P-1684, Laboratory for Information and Decision Systems, Massachusetts Institute of Technology (July 1987).
[11] S.D. O'Young and B.A. Francis, Sensitivity tradeoffs for multivariable plants, *IEEE Trans. Automat. Control* **30** (7) (July 1985) 626–632.
[12] W. Rudin, *Real and Complex Analysis* (McGraw-Hill, New York, 1987).
[13] M.S. Verma, Coprime factorizational representations and stability of nonlinear feedback systems, *Internat. J. of Control* **48** (1988) 897–918.
[14] M. Vidyasagar, Feedback stability: Necessary and sufficient conditions, *IEEE Trans. Circuits and Systems* **25** (11) (1978).
[15] J.C. Willems, *The Analysis of Feedback Systems* (MIT Press, Cambridge, MA, 1971).
[16] G. Zames and D. Bensoussan, Multivariable feedback, sensitivity, and decentralized control, *IEEE Trans. Automat. Control* **28** (11) (1983) 1030–1035.