ELSEVIER

Systems & Control Letters 24 (1995) 13–17

SYSTEMS & CONTROL LETTERS

Robust stability analysis of time-varying systems using time-varying quadratic forms*

Jeff S. Shamma

Department of Aerospace Engineering and Engineering Mechanics, The University of Texas at Austin, Austin, TX 78712, USA

(Received 13 September 1993; revised 8 January 1994)

Abstract

We consider the robust stability of time-varying linear systems subject to structured time-varying uncertainty. We provide an alternate proof of the result that robust stability holds if and only if a scaled small-gain condition holds asymptotically. The new proof employs an extension of the so-called S-procedure losslessness theorem to time-varying quadratic forms which is of independent interest.

Keywords: Robust stability; Structured uncertainty; Time-varying systems; Quadratic forms

1. Introduction

Many feedback systems with uncertain components may be transformed to the block diagram of Figure 1 (cf. [2]). In this figure, the block M represents a known stable linear system, and the block Δ represents an unknown stable linear system which belongs to a certain family of systems with a known structure and norm bound (cf. [1] and references therein).

In this paper, we consider the case of a stable linear system, M, subject to a class of stable linear time-varying perturbations, Δ. In case M is time invariant, [4] shows that robust stability holds if and only if M satisfies a scaled small-gain condition. The main tool in [4] is a so-called S-procedure losslessness theorem for time-invariant quadratic forms. In case M is time varying, [6] uses very different methods to show that an *asymptotic* scaled small-gain condition is still necessary for robust stability. Ref. [6] uses an inductive proof along with properties of the structured singular value (e.g. [5]).

In this paper, we extend the S-procedure losslessness theorem to the case of time-varying quadratic forms. Using this extension, which is of independent interest, we provide an alternate and more direct proof of the main result in [6].

2. Notation

For $v \in \mathbb{R}^n$, $(v)_i$ denotes the ith component, and $|v|$ denotes the Euclidean norm $(v^{\mathrm{T}}v)^{1/2}$. Let ℓ^2_n denote the usual space of square summable sequences in $\mathbb{R}^n$ with norm $\|\cdot\|$. The ith component of $f \in \ell^2_n$ is denoted $(f)_i$. Let $\langle\cdot,\cdot\rangle$ denote the usual inner product in ℓ^2. For $T \in \mathscr{Z}^+$, S_T denotes the T-shift (time-delay) operator and P_T denotes the truncation operator. The set of all linear causal stable q-input/p-output operators is denoted $\mathscr{L}^{p\times q}_{\mathrm{TV}}$.

---

* Supported by grants from Eglin Air Force Base, NSF grants #ECS-9296074 and #ECS-9258005, and a University Research Institute Summer Research Award.

SSDI 0167-6911(94)00010-S

3. A time-varying S-procedure losslessness theorem

In this section, we extend the so-called S-procedure losslessness theorem (cf. [3, 4]) to time-varying quadratic forms.

A version of the S-procedure losslessness theorem of [3] may be stated as follows. Let $\sigma_1(\cdot,\cdot),\dots,\sigma_N(\cdot,\cdot)$ be N quadratic forms on ℓ_n^2. Denote $\sigma_i(f,f)$ by $\sigma_i(f)$. A quadratic form is called *time-invariant* if given any $f, g \in \ell_n^2$,

$$\sigma_i(f,g) = \sigma_i(S_T f, S_T g), \quad \forall T \in \mathscr{T}^+.$$

The S-procedure losslessness theorem relates the following two conditions:

Condition A_{TI}. Given any $f \in \ell_n^2$, $\sigma_i(f) \leqslant 0$ for at least one $i \in \{1,\dots,N\}$.

Condition B_{TI}. There exist $\alpha_i > 0$, $i = 1,\dots,N$, such that for all $f \in \ell_n^2$, $\sum_{i=1}^N \alpha_i\sigma_i(f) \leqslant 0$.

Assumption 3.1. *Given any $i^* \in \{1,\dots,N\}$, there exists an $f^* \in \ell_n^2$ such that*

$$\sigma_j(f^*) > 0, \quad \forall j \neq i^*.$$

Theorem 3.1 (Megretski and Treil [4]). *Let $\sigma_1(\cdot,\cdot),\dots,\sigma_N(\cdot,\cdot)$ be time-invariant quadratic forms on ℓ_n^2 which satisfy Assumption 3.1. Then Conditions A_{TI} and B_{TI} are equivalent.*

The primary interest of Theorem 3.1 is that if the N quadratic forms cannot have positive values simultaneously, then a weighted combination of the quadratic forms is negative semi-definite.

In this section, we relax the time-invariance assumption on the quadratic forms. First, we state the following time-varying versions of Conditions A_{TI} and B_{TI} and Assumption 3.1.

Condition A_{TV}. Given any $\varepsilon > 0$, there exists a $T \in \mathscr{T}^+$ such that for all $f \in \ell_n^2$,

$$\sigma_i(S_T f) \leqslant \varepsilon \| f \|^2$$

for at least one $i \in \{1,\dots,N\}$ (which may depend on f).

Condition B_{TV}. There exist $\alpha_i > 0$ such that given any $\delta > 0$, there exists a $T \in \mathscr{T}^+$ such that for all $f \in \ell_n^2$,

$$\sum_{i=1}^N \alpha_i \sigma_i(S_T f) \leqslant \delta \| f \|^2.$$

The primary difference between these statements and their time-invariant counterparts is that the desired properties need only occur *asymptotically*.

Assumption 3.2. *There exist $b > 0$ and $\varepsilon > 0$ such that for all $T \in \mathscr{T}^+$, given any $i^* \in \{1,\dots,N\}$, there exists an $f \in \ell_n^2$ such that $\| f \| \leqslant b$, and*

$$\sigma_j(S_T f) > \varepsilon, \quad \forall j \neq i^*.$$

Similar to Assumption 3.1, Assumption 3.2 also states that any $N-1$ quadratic form can be made simultaneously positive. Two differences, however, are that this condition is required to hold over all time and that it cannot require arbitrarily large elements of ℓ_n^2.

We now state the main result of this section.

Theorem 3.2. *Let $\sigma_1(\cdot,\cdot),\dots,\sigma_N(\cdot,\cdot)$ be (possibly time-varying) quadratic forms on ℓ_n^2 which satisfy Assumption 3.2. Then Conditions A_{TV} and B_{TV} are equivalent.*

That B_{TV} implies A_{TV} follows immediately. The remainder of this section is devoted to prove A_{TV} implies B_{TV}.

Given any $T \in \mathscr{T}^+$, let $K_T \subset \mathbb{R}^N$ be defined as follows:

$$K_T \overset{\mathrm{def}}{=} \left\{ \begin{pmatrix} \sigma_1(S_T f) \\ \vdots \\ \sigma_N(S_T f) \end{pmatrix} : f \in \ell_n^2 \right\}.$$

Define $K_\infty \subset \bigcap_{T \in \mathscr{T}^+} \overline{K_T}$ as follows. For any $v \in K_\infty$, there exists a $b > 0$ (which may depend on v) such that for all $T \in \mathscr{T}^+$, there exists a sequence $\{f_n\} \subset \ell_n^2$ with $\| f_n \| \leqslant b$ such that

$$\lim_{n\to\infty} \begin{pmatrix} \sigma_1(S_T f_n) \\ \vdots \\ \sigma_N(S_T f_n) \end{pmatrix} = v.$$

The only difference between K_∞ and $\bigcap_{T \in \mathscr{T}^+} \overline{K_T}$ is the a priori bound the elements of ℓ_n^2 used in forming the sequential closure.

We first show that K_∞ is convex.

Let $v, w \in K_\infty$, and let b_v and b_w be the associated bounds in the definition of K_∞. For any $T_1 \in \mathscr{T}^+$, there exists a sequence $\{f_n^{T_1}\} \subset \ell_n^2$ be such that $\|f_n^{T_1}\| \leqslant b_v$ and

$$\lim_{n\to\infty} \begin{pmatrix} \sigma_1(S_{T_1} f_n^{T_1}) \\ \vdots \\ \sigma_N(S_{T_1} f_n^{T_1}) \end{pmatrix} = v.$$

Similarly, for any $T_2 \in \mathscr{T}^+$, there exists a sequence $\{g_m^{T_2}\} \subset \ell_n^2$ such that $\|g_m^{T_2}\| \leqslant b_w$ and

$$\lim_{m\to\infty} \begin{pmatrix} \sigma_1(S_{T_2} g_m^{T_2}) \\ \vdots \\ \sigma_N(S_{T_2} g_m^{T_2}) \end{pmatrix} = w.$$

Now define

$$h_{n,m}^{T_1,T_2} = \frac{1}{\sqrt{2}} S_{T_1} f_n^{T_1} + \frac{1}{\sqrt{2}} S_{T_2} g_m^{T_2}.$$

Note that the $h_{n.m}^{T_1,T_2}$ are uniformly bounded in ℓ_n^2. Furthermore,

$$\sigma_i(h_{n.m}^{T_1,T_2}) = \sigma_i(\tfrac{1}{\sqrt{2}} S_{T_1} f_n^{T_1}) + \sigma_i(\tfrac{1}{\sqrt{2}} S_{T_2} g_m^{T_2}) \\ + 2\sigma_i(\tfrac{1}{\sqrt{2}} S_{T_1} f_n^{T_1}, \tfrac{1}{\sqrt{2}} S_{T_2} g_m^{T_2}).$$

Given any $\varepsilon > 0$, pick n^* sufficiently large so that $|\sigma_i(\frac{1}{\sqrt{2}} S_{T_1} f_{n^*}^{T_1}) - \frac{1}{2}(v)_i| < \varepsilon$. Now pick $T_2^* > T_1$ sufficiently large so that

$$\|(I - P_{T_2^* - 1}) S_{T_1} f_{n^*}^{T_1}\| < \varepsilon.$$

Then pick m^* sufficiently large so that $|\sigma_i(\frac{1}{\sqrt{2}} S_{T_2^*} g_{m^*}^{T_2^*}) - \frac{1}{2}(w)_i| < \varepsilon$. Evaluating $h_{n^*,m^*}^{T_1,T_2^*}$ shows that

$$|h_{n^*,m^*}^{T_1,T_2^*} - (\tfrac{1}{2}(v)_i + \tfrac{1}{2}(w)_i)| \leqslant (2 + b_w)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows that $v, w \in K_\infty$ implies $\frac{1}{2}v + \frac{1}{2}w \in K_\infty$, which suffices to show that K_∞ is convex.

Define $Q_+ \overset{\text{def}}{=} \{v \in \mathbb{R}^N\colon (v)_i > 0\}$. We now show that $K_\infty \cap Q_+ = \emptyset$.

Let $v \in K_\infty$ and let b_v be the associated bound. Then for any $T \in \mathscr{T}^+$, let $\{f_n^T\} \subset \ell_n^2$ be such that $\| f_n^T \| \leqslant b_v$ and

$$\lim_{n\to\infty} \begin{pmatrix} \sigma_1(S_T f_n^T) \\ \vdots \\ \sigma_N(S_T f_n^T) \end{pmatrix} = v.$$

Using Condition A_{TV}, given any $\varepsilon > 0$, pick $T \in \mathscr{T}^+$ such that for each f_n^T, one can find an $i \in \{1, \ldots, N\}$ such that

$$\sigma_i(S_T f_n^T) \leqslant \varepsilon \|f_n^T\|^2 \leqslant \varepsilon b_v^2.$$

Since ε is arbitrary, it follows that at least one component satisfies $(v)_i = 0$. This shows that K_∞ and Q_+ are disjoint.

Since K_∞ and Q_+ are disjoint convex sets in $\mathbb{R}^N$, there exist constants α_i, $i = 1, \ldots, N$, such that

$$\alpha_1(v)_1 + \cdots + \alpha_N(v)_N \geqslant 0, \quad \forall v \in Q_+,$$

and

$$\alpha_1(v)_1 + \cdots + \alpha_N(v)_N \leqslant 0, \quad \forall v \in K_\infty.$$

Clearly the definition of Q_+ implies that $\alpha_i \geqslant 0$. In fact, one can use Assumption 3.2 to show $\alpha_i > 0$.

Finally, to prove Condition B_{TV}, consider the subset of K_T

$$B(K_T) \overset{\text{def}}{=} \left\{ \begin{pmatrix} \sigma_1(S_T f) \\ \vdots \\ \sigma_N(S_T f) \end{pmatrix} : f \in \ell_n^2,\ \|f\| \leqslant 1 \right\}.$$

Define

$$B(K_\infty) \overset{\text{def}}{=} \bigcap_{T \in \mathscr{T}^+} \overline{B(K_T)}.$$

Then $B(K_\infty) \subset K_\infty$. Since the $\overline{B(K_T)}$ form a nested sequence of compact sets in $\mathbb{R}^N$, one has that as $T \to \infty$,

$$\max_{v \in \overline{B(K_T)}} \min_{w \in B(K_\infty)} |v - w| \to 0.$$

Thus, given any $\delta > 0$ as in Condition B_{TV}, pick $T \in \mathscr{T}^+$ sufficiently large so that

$$\max_{v \in \overline{B(K_T)}} \min_{w \in B(K_\infty)} |v - w| \leqslant \delta / |\alpha|,$$

where $\alpha = (\alpha_1 \ldots \alpha_N)'$. Thus, $\|f\| \leqslant 1$ implies

$$\sum_{i=1}^N \alpha_i \sigma_i(S_T f) = \sum_{i=1}^N \alpha_i (v)_i + \sum_{i=1}^N \alpha_i (\sigma_i(S_T f) - (v)_i)$$

for some $v \in B(K_\infty)$ such that

$$\left| \begin{pmatrix} \sigma_1(S_T f) \\ \vdots \\ \sigma_N(S_T f) \end{pmatrix} - v \right| \leqslant \delta / |\alpha|.$$

Since $B(K_\infty) \subset K_\infty$, $\sum_{i=1}^N \alpha_i (v)_i \leqslant 0$. Hence

$$\sum_{i=1}^N \alpha_i \sigma_i(S_T f) \leqslant \sum_{i=1}^N \alpha_i (\sigma_i(S_T f) - (v)_i) \\ \leqslant |\alpha| \delta / |\alpha| \leqslant \delta.$$

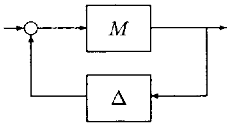

Fig. 1. Stability analysis diagram.

Since the above inequality holds for $\|f\| \leqslant 1$, it follows that

$$\sum_{i=1}^{N} \alpha_i \sigma_i(S_T f) \leqslant \delta \|f\|^2.$$

This completes the proof that A_{TV} implies B_{TV}.

4. Application to robust stability analysis

We now apply the time-varying S-losslessness result of Theorem 3.2 to the robust stability problem of Figure 1.

Let $\Delta \in \mathscr{L}_{\mathrm{TV}}^{N\times N}$ be called *admissible* if it is strictly causal and has a diagonal structure with diagonal elements $\Delta_i \in \mathscr{L}_{\mathrm{TV}}^{1\times 1}$ and $\|\Delta_i\| < 1$. Given $M \in \mathscr{L}_{\mathrm{TV}}^{N\times N}$, define $\mathbf{F}(M)$ as the family of time-varying causal operators $(I - \Delta M)^{-1}$ with Δ admissible. We will call the family $\mathbf{F}(M)$ *robustly stable* if $\mathbf{F}(M) \subset \mathscr{L}_{\mathrm{TV}}^{N\times N}$. In other words, the closed-loop system in Figure 1 is stable for all admissible Δ.

Assumption 4.1. *Let* $\mathbf{e}_i \in \mathbb{R}^n$ *denote the standard* ith *basis vector in* $\mathbb{R}^n$. *Let* $\mathbf{e}_i \in \ell^2_n$ *denote the sequence* $\mathbf{e}_i \overset{\mathrm{def}}{=} \{e_i, 0, 0, \ldots\}$. *There exists an* $\varepsilon > 0$ *such that for all* $T \in \mathscr{T}^+$, $\|(MS_T\mathbf{e}_i)_j\| > \varepsilon$, $\forall j \neq i$.

In other words, Assumption 4.1 is a non-degeneracy assumption which states that the off-diagonal terms of M are non-zero and do not asymptotically decay to zero.

Assumption 4.2. *If* $\Delta_i = 0$ *for some* $i \in \{1, \ldots, N\}$, *then* $(I - \Delta M)^{-1} \in \mathscr{L}_{\mathrm{TV}}^{N\times N}$.

Assumption 4.2 simply states that there is no destabilizing Δ with only $N - 1$ active blocks. If this is not the case, then one may delete a column and row of M to assess robust stability.

We now outline a more direct proof of the following result.

Theorem 4.1 (Shamma [6]). *Let* $M \in \mathscr{L}_{\mathrm{TV}}^{N\times N}$ *satisfy Assumptions* 4.1 *and* 4.2. *Then the family* $\mathbf{F}(M)$ *is robustly stable if and only if*

$$\inf_{T \in \mathscr{T}^+} \ \inf_{D > 0,\ \text{diagonal}} \|DMD^{-1}S_T\| \leqslant 1.$$

Proof (*sketch*). Given any $\gamma > 1$, define the quadratic forms

$$\sigma_i(f, g) \overset{\mathrm{def}}{=} \langle (Mf)_i, (Mg)_i \rangle - \gamma^2 \langle (f)_i, (g)_i \rangle, \qquad i = 1, \ldots, N.$$

With these quadratic forms, [6] shows that robust stability is equivalent to Condition A_{TV} for every $\gamma > 1$. Assumption 4.1 assures that Assumption 3.2 is satisfied. Hence, using Theorem 3.2, robust stability is equivalent to Condition B_{TV}. With the present quadratic forms, Condition B_{TV} may be stated as follows. There exist $\alpha_i > 0$ such that given any $\delta > 0$, there exists a $T \in \mathscr{T}^+$ such that for all $f \in \ell^2_N$,

$$\sum_{i=1}^{N} \alpha_i \|(MS_T f)_i\|^2 \leqslant \delta \|f\|^2 + \gamma^2 \sum_{i=1}^{N} \alpha_i \|(S_T f)_i\|^2.$$

Using

$$D = \begin{pmatrix} \sqrt{\alpha_1} & & \\ & \ddots & \\ & & \sqrt{\alpha_N} \end{pmatrix}$$

leads to $\inf_{T \in \mathscr{T}^+} \|DMD^{-1}S_T\| \leqslant \gamma$. Using successively smaller values of γ leads to the desired result. □

Acknowledgements

The author thanks Anders Rantzer and John Doyle for helpful discussions.

References

[1] P. Dorato, Ed. *Robust Control* (IEEE Press, New York, 1987).

[2] J.C. Doyle, J.E. Wall and G. Stein, Performance and robustness analysis for structured uncertainty, in: *Proc. 21st IEEE Conference on Decision and Control*, December (1982) pp. 629–636.

[3] A. Megretski, Necessary and sufficient conditions of stability: A multiloop generalization of the circle criterion. *IEEE Trans. Automat. Control* **AC-38**(5) (1993) 753–756.

[4] A. Megretski and S. Treil, Power distribution inequalities in optimization and robustness of uncertain systems. *J. Math. Systems, Estimation, Control* **3**(3) (1993) 301–319.

[5] A. Packard and J.C. Doyle, The complex structured singular value. *Automatica* **29**(9) (1993) 71 110. Special Issue on Robust Control.

[6] J.S. Shamma, Robust stability with time-varying structured uncertainty, *IEEE Trans. Automat. Control*, to appear.