Optimization of the $\ell^\infty$-Induced Norm 
Under Full State Feedback

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Abstract—This paper considers the $\ell^1$-optimal control problem, i.e., minimization of the effects of disturbances as measured by the $\ell^\infty$-induced norm. Earlier work showed that even in the case of full state feedback, optimal and near-optimal nonlinear controllers may be dynamic and of arbitrarily high order. However, previous work by the author derived the existence of near-optimal nonlinear controllers which are static. This paper presents a constructive algorithm for such nonlinear controllers. The main idea is to construct a certain subset of the state space such that achieving disturbance rejection is equivalent to restricting the state dynamics to this set. A construction of this subset requires the solution of several finite, linear programs with the number of variables being at least the number of states but less than the number of states plus the number of controls. The concept of controlled invariance plays a central role throughout.

I. INTRODUCTION

FEEDBACK control objectives such as command following, disturbance rejection, and performance robustness can often be phrased as induced norm optimization problems. Two popular paradigms are $\mathcal{H}_\infty$-optimal control introduced in [39] and $\ell^1$-optimal control introduced in [37]. The $\mathcal{H}_\infty$-optimal control problem measures a feedback system’s performance in terms of the $\ell^2$-induced norm which is the maximum possible amplification of signal energy. The $\ell^1$-optimal control problem measures performance in terms of the $\ell^\infty$-induced norm which is the maximum possible amplification of signal magnitude.

Early solutions to the $\mathcal{H}_\infty$-optimal control problem (cf., [23]) employed a parameterization of all stabilizing controllers and operator theoretic best approximation methods. These solutions were followed by so-called “state-space solutions” (cf., [21] and references therein) which employed a more direct approach having several classical optimal control interpretations (e.g., [4]). One important interpretation was an underlying separation structure. Namely, the optimal output feedback controller consists of an optimal estimate of the optimal state feedback controller. Since the optimal state feedback control is a static gain, the output feedback control is dynamic for the sake of optimal estimation.

The current solution to the $\ell^1$-optimal control problem [16], [17] also employs a parameterization of all stabilizing controllers and best approximation methods. The resulting best approximation problems are well handled by solving appropriate linear programs [20].

Reference [19] considered $\ell^1$-optimal control with full state feedback. The authors showed that unlike the $\mathcal{H}_\infty$-optimal case, optimal as well as near-optimal controllers can be dynamic and of arbitrarily high order. This result disqualifies the interpretation of dynamic controllers being solely for estimation. However, [34] derived (in a nonconstructive manner) that static nonlinear state feedback performs as well as linear dynamic feedback. In other words, full state feedback $\ell^1$-optimal control need not require dynamics if nonlinear controllers are admissible.

In this paper, we follow on the work of [34]. In particular, we provide a constructive algorithm for near-optimal nonlinear state feedback. The main idea is to show that the existence of a controller which delivers a specified level of performance is equivalent to the existence of a certain subset of the state space which can be made invariant under feedback. The method of [34] starts with a linear dynamic controller to construct an appropriate invariant set. In this paper, we construct the desired set directly by imposing a succession of inequality restrictions on the state. If the inequalities become inconsistent, then no controller exists which delivers the specified performance. However if the inequality succession remains consistent, it may be terminated to construct the desired full state feedback. This approach leads to a procedure similar to $\gamma$-iteration [21] in the computation of the optimal performance and controller. Unlike [34], we do not rely on a linear dynamic controller to construct the desired nonlinear controller.

The method of constructing controlled invariant sets has been used extensively throughout the controls literature from a variety of contexts including viability theory and differential inclusions [1], [2], [24], [31], [32], dynamic programming [5], [6], systems with control constraints [7], [8], [15], [25]–[27], [29], construction of reachable sets [12], [13], and time-varying system analysis [10], [11], [33]. Recent work in the context of optimal disturbance rejection includes [9], [22], [30], and [36].

The remainder of this paper is organized as follows. Section II presents some background material. Section III presents the problem formulation and reviews the results of reference [34]. Section IV discusses the notion of a controlled invariance kernel. Section V presents the main construction of near-optimal nonlinear state feedback. Section VI presents some examples. Finally, Section VII contains some concluding remarks.

II. MATHEMATICAL PRELIMINARIES

A. Basic Notation

First, we review some notation regarding $\ell^1$-optimal control and operator norms (cf., [16] and [38]).
Let $\mathbb{R}^+$ denote the set of nonnegative real numbers and $\mathbb{Z}^+$ denote the set of nonnegative integers. For $M \in \mathbb{R}^{m \times n}$, let $M(i,j)$ denote the $(i,j)$th element of $M$, let $M(i,:)$ denote the $i$th row of $M$, and define

$$|M(i,:)| = \sum_{j=1}^{n} |M(i,j)|$$

and

$$|M| = \max_{i} |M(i,:)|.$$  

Similarly for $x \in \mathbb{R}^n$, let $x_i$ denote the $i$th component of $x$ and define

$$|x| = \max_{i} |x_i|.$$  

The appropriate definition of $|\cdot|$ will be apparent from context.

Let $E_+^{\infty}(\mathbb{Z}^+)$ denote the set of bounded one-sided sequences in $\mathbb{R}^n$. For $f = \{f(0), f(1), f(2), \cdots \} \in E_+^{\infty}(\mathbb{Z}^+)$, define

$$\|f\| = \sup_{t \in \mathbb{Z}^+} |f(t)|.$$  

A causal operator $H: E_+^{\infty}(\mathbb{Z}^+) \rightarrow E_+^{\infty}(\mathbb{Z}^+)$ is called stable if

$$\|H\| \overset{\text{def}}{=} \sup_{f \neq 0} \frac{\|Hf\|}{\|f\|} < \infty.$$  

B. Set-Valued Mappings

We now collect some concepts and facts from set-valued analysis and viability theory following [1]-[3].

A set-valued map, denoted $F: X \rightrightarrows Y$, is a mapping from points $x \in X$ to subsets $F(x) \subset Y$. We define $\text{dom}(F) = \{x \in X: F(x) \neq \emptyset\}$.

**Definition 2.1** [1, p. 56]: Let $X$ and $Y$ be Banach spaces.

A set-valued map $F: X \rightrightarrows Y$ is called lower semicontinuous if for any $x \in \text{dom}(F)$, $y \in F(x)$, and sequence $x_n \in \text{dom}(F)$ converging to $x$, there exists a sequence of elements $y_n \in F(x_n)$ converging to $y$.

A set-valued map $F: X \rightrightarrows Y$ is called upper semicontinuous if 1) $\text{dom}(F)$ is closed, and 2) for any $x \in \text{dom}(F)$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that $x' \in \text{dom}(F)$ and $\|x' - x\| < \delta$ together imply

$$\sup_{y' \in F(x')} \inf_{y \in F(x)} \|y' - y\| < \varepsilon.$$  

An important concept is that of a single-valued function, called a selection function, within the set-valued map.

**Theorem 2.1** [1, p. 228]: Let $X$ be a compact metric space and $Y$ be a Banach space. Let $F: X \rightrightarrows Y$ be a lower semicontinuous set-valued map with closed convex values. Then there exists a continuous $f: X \rightarrow Y$ such that for all $x \in X$, $f(x) \in F(x)$.

Finally, we have the following condition for lower semicontinuity.

**Theorem 2.2** [2, p. 49]: Let $X$ be a metric space and $Y$ and $U$ be Banach spaces. Let $f: X \times U \rightarrow Y$ be a continuous map such that for all $x \in X$, $u \rightarrow f(x,u)$ is affine. Let $T: X \rightrightarrows Y$ and $C: X \rightrightarrows U$ be lower semicontinuous set-valued maps with closed convex values, and let $C$ be locally bounded. Suppose there exists an $\alpha > 0$ such that for all $x \in X$, there exists a $u \in C(x)$ such that

$$f(x,u) + r \in T(x), \quad \forall \|r\| \leq \alpha.$$  

Then the set-valued $R: X \rightrightarrows U$ defined by

$$R(x) = \{u \in C(x): f(x,u) \in T(x)\}$$  

is lower semicontinuous.

C. Projections of Convex Sets

Let $1$ denote a column vector of appropriate length with unit elements. For $M \in \mathbb{R}^{m \times n}$, let $\text{Set}(M)$ denote the subset of $\mathbb{R}^n$ associated with $M$ defined by the constraints

$$\text{Set}(M) = \{x: Mx \leq 1\}.$$  

Let $S$ be a subset of $\mathbb{R}^{n+1}$ characterized by $S = \text{Set}(M)$. Partition $M = (M_1 \ M_2)$ where $M_1$ has $n$ columns. Then

$$\text{Set}(M) = \{(v,w) \in \mathbb{R}^n \times \mathbb{R}: (M_1 \ M_2) \begin{pmatrix} v \\ w \end{pmatrix} < 1\}.$$  

Let $\tilde{S} = \{v \in \mathbb{R}^n: (v,w) \in \text{Set}(M) \text{ for some } w \in \mathbb{R}\}$. We would like to find a matrix $\tilde{M}$ such that $\tilde{S} = \text{Set}(\tilde{M})$. In doing so, we rephrase the original constraints on $n+1$ variables as new constraints on the first $n$ variables.

**Definition 2.2**: Let $M \in \mathbb{R}^{n \times (n+1)}$. Define $\text{Rack}[M]$ as the set of matrices, $\tilde{M}$, such that

$$v \in \text{Set}(\tilde{M}) \subset \mathbb{R}^n$$

$$\iff$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \text{Set}(M) \subset \mathbb{R}^{n+1},$$

for some $w \in \mathbb{R}$.

**Proposition 2.1**: Let $M = (M_1 M_2)$, where $M_1$ has $n$ columns and $M_2$ has one column. Let

$$Z_+ = \{i: (M_2)_i > 0\}$$

$$Z_- = \{i: (M_2)_i < 0\}$$

$$Z_0 = \{i: (M_2)_i = 0\}.$$  

Let $\tilde{M}$ be the matrix formed by the rows:

- $\tilde{M}_+ = \begin{pmatrix} (M_1)_{i+:} - (M_2)_{i+:} \\ (M_1)_{i+:} \end{pmatrix}, \quad \forall i_+ \in Z_+, i_- \in Z_-.$  

- $\tilde{M}_0 = (M_1)_{i_0}, \quad \forall i_0 \in Z_0.$  

Then $\tilde{M} \in \text{Rack}[M]$.

**Proposition 2.2**: In the framework of Proposition 2.1, let $Z_+$ and $Z_-$ be nonempty. Define $\phi_+^M: \mathbb{R}^n \rightarrow \mathbb{R}$ and $\phi_-^M: \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\phi_+^M(v) = \min_{i_+ \in Z_+} \frac{1 - (M_1)_{i_+:} v^{i_+}}{(M_2)_{i_+:}}$$

$$\phi_-^M(v) = \max_{i_- \in Z_-} \frac{1 - (M_1)_{i_-:} v^{i_-}}{(M_2)_{i_-:}}.$$  

Then $\phi_+^M$ and $\phi_-^M$ are continuous and satisfy

$$\begin{pmatrix} v, \phi_+^M(v) \end{pmatrix} \in \text{Set}(M), \quad \forall v \in \text{Set}(\tilde{M})$$

$$\begin{pmatrix} v, \phi_-^M(v) \end{pmatrix} \in \text{Set}(M), \quad \forall v \in \text{Set}(\tilde{M}).$$
The proofs for Propositions 2.1 and 2.2 follow the development of the Fourier–Motzkin algorithm which is described in [29]. Note that the construction in Proposition 2.1 typically leads to several redundant constraints which may be removed by solving appropriate linear programs. The functions \( \phi^M_w \) and \( \phi^M_u \) in Proposition 2.2 are simply explicitly constructed continuous selection functions of the set-valued map

\[ v \mapsto \{ w : (u, w) \in \operatorname{Set}(M) \}. \]

Any convex combination \( \alpha \phi^M_w + (1 - \alpha) \phi^M_u \) with \( \alpha \in [0, 1] \) is also a continuous selection.

III. PROBLEM FORMULATION

The system dynamics under consideration are

\[
\begin{align*}
    x(t+1) &= Ax(t) + B_1w(t) + B_2u(t) \\
    z(t) &= C_1x(t) + D_{11}w(t) + D_{12}u(t) \\
    y(t) &= z(t).
\end{align*}
\]

The vector signals are defined as follows: \( x \in \mathbb{R}^n \) denotes the state, \( w \in \mathbb{R}^m \) denotes exogenous disturbances, \( u \in \mathbb{R}^m \) denotes the control input, \( z \in \mathbb{R}^p \) denotes regulated outputs, and \( y \in \mathbb{R}^q \) denotes measured outputs.

The following controller forms are called admissible:

1. Linear dynamic feedback (\( K_{dy} \))

\[
x_K(t+1) = A_K x_K(t) + B_K u(t) \\
    u(t) = C_K x_K(t) + D_K y(t).
\]

2. Nonlinear static feedback (\( K_{st} \))

\[
u(t) = g(y(t)).
\]

where \( g \) is continuous and \( g(0) = 0 \).

Given an admissible controller, \( K \), define \( T_{zw}(K) \) to be the forced dynamics from \( w \) to \( z \) with zero initial conditions. Similarly define \( T_{zu}(K) \) and \( T_{wu}(K) \).

**Definition 3.1:** An admissible controller, \( K \), is said to be internally stabilizing with a performance (respectively, strict performance) of \( \gamma \) if 1) the unforced dynamics \( (w = 0) \) are globally exponentially stable and 2) the forced dynamics with zero initial conditions satisfy \( \|T_{zw}(K)\| \leq \gamma \) (respectively, \( \|T_{zw}(K)\| < \gamma \)) with both \( \|T_{zu}(K)\|, \|T_{wu}(K)\| < \infty \).

The optimal performance, denoted \( \gamma_{opt} \), is defined as

\[
\gamma_{opt} = \inf_{K} \{ \|T_{zw}(K)\| : K \text{ is admissible and internally stabilizing} \}.
\]

The following result motivates the present investigation of nonlinear static feedback.

**Theorem 3.1 [34]:** There exists an internally stabilizing linear dynamic controller, \( K_{dy} \), with a strict performance of \( \gamma \) only if there exists an internally stabilizing continuous static nonlinear controller, \( K_{st} \), with a strict performance of \( \gamma \).

The proof in [34] assumes the existence of a linear dynamic controller to derive a particular subset of the state space which can be made invariant under feedback. It turns out that invariance of this subset implies achieving the desired performance. Reference [34] goes on to show that a nonlinear static feedback must also exist which makes the subset invariant and hence achieves the desired performance. Reference [35] shows that nonlinear controllers can in fact outperform linear controllers.

Section IV next provides an algorithm which shows—at least conceptually—that the initial step of assuming the existence of any previous controller may be bypassed. Section V goes on to construct the desired state feedback.

IV. CONTROLLED INVARIANCE KERNELS

In this section we define a notion of controlled invariance for difference inclusions. This notion will prove central to the construction of nonlinear static state feedback. The present discussion employs the language of viability theory [1] for differential inclusions. However, as mentioned in the introduction, similar methods have been used in a variety of different contexts.

We first consider the difference inclusion

\[
x(t+1) \in F(x(t))
\]

where \( F : \mathbb{R}^n \to \mathbb{R}^n \).

**Definition 4.1:** Let \( F : \mathbb{R}^n \to \mathbb{R}^n \). A subset \( K \subset \operatorname{dom}(F) \) is invariant under \( F \) if \( F(x) \subset K \) for every \( x \in K \).

The implications of invariance for the difference inclusion (2) are, of course, that if the solution starts in \( K \), it remains in \( K \). We also will use the expression “\( K \) is invariant under the difference inclusion (2).”

**Definition 4.2:** Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) and \( K \subset \operatorname{dom}(F) \). The largest closed subset of \( K \) which is invariant under \( F \) is the invariance kernel of \( K \) for \( F \) and is denoted \( \operatorname{Inv}(K) \).

Under certain conditions the invariance kernel, if it exists, may be constructed through the so-called Invariance Kernel algorithm. A related algorithm for differential inclusions is the viability kernel algorithm which has been applied to control systems in the form of the “zero-dynamics algorithm” [28] for nonlinear systems.

**Proposition 4.1 (Invariance Kernel Algorithm [1]):** Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be lower semicontinuous and let \( K \subset \operatorname{dom}(F) \) be closed. Define recursively the subsets \( K_j \) by

\[
K_0 = K \\
K_{j+1} = \{ x \in K_j : F(x) \subset K_j \}.
\]

Then

\[
\operatorname{Inv}(K) = \bigcap_{j=0}^{\infty} K_j.
\]

**Proof:** We first show that \( K_{j+1} \) is closed if \( K_j \) is closed. Let \( \{x_n\} \) be a sequence in \( K_{j+1} \), which converges to \( x_0 \), \( x_0 \in K_{j+1} \). By lower semicontinuity, for any \( y_0 \in F(x_0) \), there exist \( y_n \in F(x_n) \subset K_j \), which converge to \( y_0 \), \( y_n \in K_j \) since \( K_j \) is closed, and hence \( K_{j+1} \) is closed. Since \( K_0 \) is closed, all of the \( K_j \) are closed as well as \( \bigcap_{j=0}^{\infty} K_j \) is closed.

Clearly \( \operatorname{Inv}(K) \), if it exists, is contained in \( \bigcap_{j=0}^{\infty} K_j \). Since \( x \in \bigcap_{j=0}^{\infty} K_j \) implies \( F(x) \subset \bigcap_{j=0}^{\infty} K_j \), \( \bigcap_{j=0}^{\infty} K_j \) is invariant under \( F \) which completes the proof.

We now define the notion of a controlled difference inclusion.
Definition 4.3: Let $F: \mathcal{R}^n \rightarrow \mathcal{R}^n$ and $U: \mathcal{R}^n \rightarrow \mathcal{R}^n$ be set-valued maps and $f: \mathcal{R}^n \times \mathcal{R}^m \rightarrow \text{dom}(F)$ be a single-valued map. Define

$$\bar{F}(x) = \bigcup_{u \in U(x)} F(f(x, u)).$$

Then

$$x(t+1) \in \bar{F}(x)$$

is the controlled difference inclusion defined by $(F, U, f)$.

Definition 4.4: Consider a controlled difference inclusion defined by $(F, U, f)$. A subset $K \subset \text{dom}(U)$ is controlled invariant under $(F, U, f)$ if for every $x \in K$, there exists an $u \in U(x)$ such that $F(f(x, u)) \subset K$.

Note that this definition intentionally does not specify any particular feedback law.

Definition 4.5: Consider a controlled difference inclusion defined by $(F, U, f)$. Let $K \subset \text{dom}(U)$. The largest closed subset of $K$ which is controlled invariant under $(F, U, f)$ is the controlled invariance kernel of $K$ for $(F, U, f)$ and is denoted $\text{CInv}(K)$.

The controlled invariance kernel, if it exists, may be obtained under certain conditions using the Controlled Invariance Kernel algorithm.

Definition 4.6: We say the controlled difference inclusion defined by $(F, U, f)$ has the locally bounded control property if either:

1) $U$ is locally bounded, or
2) a) $x \in F(\xi)$, for all $\xi \in \text{dom}(F)$;
   b) $|u| \rightarrow \infty$ implies $|f(x, u)| \rightarrow \infty$, for all $x \in \mathcal{R}^n$.

The implication of the locally bounded control property is an effective bound (either explicit or implicit) on the admissible control.

Proposition 4.2 (Controlled Invariance Kernel Algorithm): Let the controlled difference inclusion defined by $(F, U, f)$ have the locally bounded control property. Assume further that $\text{dom}(F)$ is closed, $F$ is lower semicontinuous, $f$ is continuous, and $U$ is upper semicontinuous and closed valued. Let $K \subset \text{dom}(U)$ be compact. Define recursively the subsets $K_j$ and $K_{j+1}$ by

$$K_0 = K,$$

$$K_{j+1} = \{\xi \in \text{dom}(F): F(\xi) \subset K_j\},$$

for some $u \in U(x)$. Then $\text{CInv}(K) = \bigcap_{j=0}^{\infty} K_j$.

Proof: Using arguments similar to the proof of Proposition 4.1, we can show that if $K_j$ is closed, then $K_{j+1}$ is closed. We now show that if $K_{j+1}$ is closed, then $K_{j+1+1}$ is closed. Let $\{x_n\} \subset K_{j+1}$ be a sequence which converges to $x_0 \in K_j$. Let $u_n \in U(x_n)$ be such that $f(x_n, u_n) \in K_{j+1}$. From the locally bounded control property and the upper semicontinuity of $U$, the set $\{u \in U(x): x \in K_{j+1}\}$ is bounded. Therefore, a subsequence which we label $u_n$ converges to some $u_0$. Since $U$ is upper semicontinuous with closed values, $u_0 \in U(x_0)$. By continuity of $f$, the $f(x_n, u_n) \in K_{j+1}$ converge to $f(x_0, u_0)$. Since $K_{j+1}$ is closed, we see that $f(x_0, u_0) \in K_{j+1}$ which implies $x_0 \in K_{j+1}$, and hence $K_{j+1}$ is closed.

Clearly $\text{CInv}(K) \subset \bigcap_{j=0}^{\infty} K_j$ if it exists. Since the $K_j$ are nested compact sets, $\bigcap_{j=0}^{\infty} K_j$ is empty if and only if $K_j$ is empty for some $j$. In this case, the proposition holds trivially.

In case $\bigcap_{j=0}^{\infty} K_j$ is nonempty, we will show it is controlled invariant. Define the regulation maps $R_0(x) = U(x)$ and

$$R_{j+1}(x) = \{u \in U(x): f(x, u) \in K_{j+1}\}.$$

Using the locally bounded control property, that $f$ is continuous, and that $K_{j+1}$ is closed, we have $R_{j+1}(x)$ compact. Since $K_j$ is nonempty, so is $R_0(x)$ for all $x \in K_j$. Therefore $\bigcap_{j=0}^{\infty} R_j(x)$ is nonempty for every $x \in \bigcap_{j=0}^{\infty} K_j$. Then $x \in \bigcap_{j=0}^{\infty} K_j$ and $u \in \bigcap_{j=0}^{\infty} R_j(x)$ implies $f(x, u) \subset \bigcap_{j=0}^{\infty} K_j$ which completes the proof.

Note that the set $K_j$ has the simple interpretation of all initial states for which it is possible to remain inside $K$ for $j$ time steps.

We see that the Controlled Invariance Kernel algorithm not only determines whether invariance is possible within a specified set but also what values of the control must be applied. These control values may be generated through dynamic or static feedback. However, since the necessary control values are a function of the current state, it becomes apparent that static state feedback, when available, suffices for invariance purposes.

Reference [27] presents a related algorithm for constructing maximal controlled invariant subsets for linear difference equations. The method in [27] is to construct the set of reachable states starting from the origin, subject to control and state constraints. This is done recursively by constructing successively larger subsets of the state space, as opposed to Proposition 4.2 which constructs successively smaller subsets. The latter approach seems more amenable to controlled systems with disturbances.

V. CONSTRUCTION OF NONLINEAR STATE FEEDBACK

A. Standing Assumptions

We make the following assumptions on (1) for the remainder of the paper.

Assumption 5.1:

1) $B_1$ and $C_1$ have rank $n$.
2) $D_{12}^2 C_1 = 0$, $D_{12}^2 D_{12} > 0$.
3) $B_2$ has rank $m$.

Statements 1) and 2) are mildly restrictive but offer considerable simplification. Statement 3) implies there are no control redundancies.

B. Relating Controlled Invariance and Disturbance Rejection

In this section, we relate the disturbance rejection problem for (1) to the existence of nonempty controlled invariance kernels.
We may write (1) as a controlled difference inclusion as follows. For \( \gamma > 0 \), define \((F_\gamma, U_\gamma, f)\) by
\[
F_\gamma(w) = \{ \xi + \frac{1}{\gamma} B_1 w \in \mathbb{R}^n : |w| \leq 1 \}
\]
\[
U_\gamma(x) = \{ u \in \mathbb{R}^m : C_1 x + \frac{1}{\gamma} D_{11} w + D_{12} u \leq 1, \forall |w| \leq 1 \}
\]
\[f(x,u) = Ax + B_2 u.
\] (3)

Then the controlled difference inclusion defined by \((F_\gamma, U_\gamma, f)\) in (3) satisfies the hypotheses of Proposition 4.2 whenever \(\text{dom}(U_\gamma)\) is nonempty.

**Proposition 5.1:** The controlled difference inclusion defined by \((F_\gamma, U_\gamma, f)\) in (3) satisfies the hypotheses of Proposition 4.2 whenever \(\text{dom}(U_\gamma)\) is nonempty.

**Proof (Sketch):** Clearly \(F_\gamma\) is lower semicontinuous, \(f\) is continuous, and \(U_\gamma\) is closed-valued. Using that \(B_2\) has rank \(m\) leads to the locally bounded control property. We still must show that \(U_\gamma\) is upper semicontinuous. This can be accomplished by the orthogonal decomposition \(u = u_6 + u_6\) where \(u_6\) is in the null space of \(D_{12}\). Simply exploit that \(|D_{12} u_6| \geq \alpha |u_6|\) for some \(\alpha > 0\).

**Theorem 5.2:**

\[
\gamma_{opt} = \inf \{ \gamma : \text{CINV}(\text{dom}(U_\gamma)) \text{ is nonempty} \}
\]

The following lemma will be useful.

**Lemma 5.1:** If \(\text{CINV}(\text{dom}(U_\gamma))\) is nonempty, it has the following properties: 1) symmetry with respect to the origin, 2) convexity, and 3) nonempty interior containing the origin.

**Proof (Lemma 5.1):** Clearly \(\text{dom}(U_\gamma)\) exhibits these properties for \(\gamma > \gamma_{opt}\). Let \(K_\gamma\) and \(K_\gamma^{1/2}\) be as defined in the Controlled Invariance Kernel algorithm for \(\text{dom}(U_\gamma)\). A simple recursive argument on these sets leads to Properties 1) and 2). To show Property 3), first note that \(0 \in \text{CINV}(\text{dom}(U_\gamma))\). Let \(u \in \text{CINV}(\text{dom}(U_\gamma))\). Then \(F_\gamma(-B_2 u) = -F_\gamma(B_2 u) \subset \text{CINV}(\text{dom}(U_\gamma))\). Since \(B_1\) has rank \(n\), both sets have nonempty interior. Since \(\text{CINV}(\text{dom}(U_\gamma))\) is convex, the set \(\frac{1}{2} F_\gamma(B_2 u) + \frac{1}{2} F_\gamma(-B_2 u) \subset \text{CINV}(\text{dom}(U_\gamma))\) contains the origin and has nonempty interior.

**Proof of Theorem 5.1:** Let \(K\) be an internally stabilizing controller which achieves a performance of \(\gamma\). We will show \(\text{CINV}(\text{dom}(U_\gamma))\) is nonempty.

Let \(S\) denote the set of reachable plant states under closed-loop operation with zero initial conditions and \(|u| \leq 1\). We will show that the closure, \(\bar{S}\), is a controlled invariant set. Since the controlled difference inclusion for \((F_\gamma, U_\gamma, f)\) in (3) satisfies the hypotheses of Proposition 4.2, \(\text{CINV}(\text{dom}(U_\gamma))\) is possibly empty but always exists. Hence if \(\bar{S}\) is a controlled invariant set, then \(\text{CINV}(\text{dom}(U_\gamma))\) is nonempty.

Toward this end, let
\[S_{1/2} = \{ \xi : \xi + \frac{1}{\gamma} B_1 w \in \bar{S}, \forall |w| \leq 1 \}.
\]
We see that \(S_{1/2}\) is closed. Let
\[R(x) = \{ u \in U_\gamma(x) : Ax + B_2 u \in S_{1/2} \}.
\]
For \(x_0 \in \bar{S}\), let \(\{ x_n \} \subset S\) converge to \(x_0\). The existence of \(K\) assures \(R(x)\) is nonempty for every \(x \in \bar{S}\). Thus let \(u_0\) be such that \(Ax_0 + B_2 u_0 \in S_1\). Since \(\{ u_n \}\) must be bounded, there exists a subsequence, which we relabel \(u_n\), which converges to some \(u_0\). Using that \(\{ Ax_n + B_2 u_n \} \subset S_1\) converges to \(Ax_0 + B_2 u_0\) and the fact that \(S_1\) is closed shows \(u_0 \in R(x_0)\).

Thus for any \(x \in \bar{S}\), there exists a \(u_0 \in U_\gamma(x_0)\) such that
\[Ax_0 + B_2 u_0 + \frac{1}{\gamma} B_1 w \in \bar{S}, \forall |w| \leq 1.
\]
This shows that \(\bar{S}\) is a controlled invariant set as desired, and hence \(\text{CINV}(\text{dom}(U_\gamma))\) is nonempty.

Now assume that \(\text{CINV}(\text{dom}(U_\gamma))\) is nonempty. For any \(\varepsilon > 0\), we will construct a controller \(K_{\text{opt}}\) which achieves a performance of \(\gamma + \varepsilon\).

Define
\[K_{\infty, \frac{1}{2}} = \{ \xi : \xi + \frac{1}{\gamma} B_1 w \in \text{CINV}(\text{dom}(U_\gamma)) \}
\]
\[\forall |w| \leq \frac{\gamma}{\gamma + \varepsilon}.
\]
Note the extra restriction on \(w\). Redefine
\[R(x) = \{ u \in U_\gamma(x) : Ax + B_2 u \in K_{\infty, \frac{1}{2}} \}.
\]
The interpretation of \(R(x)\) is similar to its original definition earlier in the proof. Namely, \(R(x)\) characterizes the set of admissible control values for controlled invariance. We see that \(R(x)\) is convex for any \(x \in \text{CINV}(\text{dom}(U_\gamma))\). Furthermore, because of the extra restriction on \(w\), there exists an \(\alpha\) such that for any \(x \in \text{CINV}(\text{dom}(U_\gamma))\), there exists a \(u \in R(x)\) such that \(Ax + B_2 u + r \in K_{\infty, \frac{1}{2}}\) for all \(|r| \leq \alpha\). A straightforward application of Theorem 2.2 shows that \(R(x)\) is lower semicontinuous on \(\text{CINV}(\text{dom}(U_\gamma))\). Since \(R(x)\) has convex values on \(\text{CINV}(\text{dom}(U_\gamma))\), there exists a continuous selection function denoted \(\tilde{g}\).

This construction of \(\tilde{g}\) assures that \(\text{CINV}(\text{dom}(U_\gamma))\) is invariant under the difference inclusion
\[x(t + 1) \in \{ Ax(t) + B_2 \tilde{g}(x(t)) + \frac{1}{\gamma} B_1 w : |w| \leq \frac{\gamma}{\gamma + \varepsilon} \}.
\]

We now employ a scaling argument used in [34] which leads to a state feedback defined on the entire state space. Define
\[p : \mathbb{R}^n \to \mathbb{R}^+ \text{ by } p(x) = \inf \{ \alpha \in \mathbb{R}^+ : x \in \alpha \text{ CINV}(\text{dom}(U_\gamma)) \}.
\]
Since \(\text{CINV}(\text{dom}(U_\gamma))\) has the properties in Lemma 5.1, \(p\) defines a norm on \(\mathbb{R}^n\) [14, p. 106]. Now define \(g : \mathbb{R}^n \to \mathbb{R}\) as follows. Set \(g(0) = 0\) and
\[g(x) = p(x) \tilde{g}(x/p(x)).
\]
Note that the definition of \(g\) only uses \(\tilde{g}\) defined on the boundary of \(\text{CINV}(\text{dom}(U_\gamma))\). The arguments in [34] show that this feedback is internally stabilizing with a performance of \(\gamma + \varepsilon\).

It is important to note that whether \(\text{CINV}(\text{dom}(U_\gamma))\) is nonempty or not depends on the system dynamics only and not on any particular controller structure. In fact, the theorem
would hold even if we allow discontinuous nonlinear dynamic feedback.

The proof of Theorem 5.1 suggests that an implementation of the Controlled Invariant Kernel algorithm can be used both to compute \( \gamma_{\text{opt}} \) as well as compute near-optimal controllers as follows:

1) Initialize \( \gamma > 0 \) and \( K_0 = \text{dom}(U_\gamma) \).
2) Execute the Controlled Invariant Kernel algorithm to compute \( K_j \) and \( K_{j+\frac{1}{2}} \):
   a) If any \( K_j \) is empty, then \( \text{CINV}(\text{dom}(U_\gamma)) \) is empty. Increase \( \gamma \) and restart.
   b) If \( \text{CINV}(\text{dom}(U_\gamma)) \) is nonempty:
      i) Decrease \( \gamma \) and restart; or
      ii) Construct an internally stabilizing \( K_{st} \) which achieves a performance of \( \gamma + \epsilon \) with \( \epsilon > 0 \) arbitrary.

Two problems with the above procedure follow. The first problem is that the proof of Theorem 5.1 does not provide an explicit construction of a controller \( K_{st} \). The second problem is that it is unclear when to terminate computing the \( K_j \). Although the algorithm will finitely-terminate if \( \text{CINV}(\text{dom}(U_\gamma)) \) is empty, we may not be able to provide a bound on how many iterations are sufficient.

These issues are addressed in the following sections.

**C. Algorithm Implementation: Scalar Control**

We now provide an explicit implementation of the Controlled Invariant Kernel algorithm to construct \( \text{CINV}(\text{dom}(U_\gamma)) \) when \( \nu \) is a scalar. The implementation will terminate whenever \( \gamma \leq \gamma_{\text{opt}} \).

\( D_{11}, D_{12} = 0 \): Substituting \( D_{11} = 0 \) and \( D_{12} = 0 \) into (3) shows that

\[
U_\gamma(x) = \{ u \in \mathcal{R} : |C_1 x| \leq 1, \quad \forall |w| \leq 1 \}.
\]

In this case

\[\text{dom}(U_\gamma) = \{ x : |C_1 x| \leq 1 \}\]

and \( U_\gamma(x) = \mathcal{R} \) on its domain.

To construct \( \text{CINV}(\text{dom}(U_\gamma)) \), we will compute matrices \( M_j \) and \( M_{j+\frac{1}{2}} \) so that \( K_j = \text{Set}(M_j) \) and \( K_{j+\frac{1}{2}} = \text{Set}(M_{j+\frac{1}{2}}) \).

We start by initializing \( K_0 = \text{dom}(U_\gamma) \). If we define

\[
M_0 = \begin{pmatrix} C_1 \\ -C_1 \end{pmatrix}
\]

then clearly \( \text{Set}(M_0) = K_0 = \text{dom}(U_\gamma) \).

To construct \( M_{j+1} \), we see that \( \xi \in M_{j+\frac{1}{2}} \) requires

\[
M_{j+\frac{1}{2}} \left( \xi + \frac{1}{\gamma} B_1 w \right) \leq 1, \quad \forall |w| \leq 1.
\]

Using a row-by-row analysis, we require that each row satisfies

\[
(M_0)_{(i,:)} \xi + \frac{1}{\gamma} (M_0 B_1)_{(i,:)} w \leq 1, \quad \forall |w| \leq 1.
\]

There are three possibilities for each row:

1) If \( \frac{1}{\gamma} |(M_0 B_1)_{(i,:)}| \geq 1 \), then \( K_{j+\frac{1}{2}} \) is empty and so is \( \text{CINV}(\text{dom}(U_\gamma)) \).

2) If \( \frac{1}{\gamma} |(M_0 B_1)_{(i,:)}| = 1 \), then \( \gamma \leq \gamma_{\text{opt}} \).

3) If \( \frac{1}{\gamma} |(M_0 B_1)_{(i,:)}| < 1 \), then \( \xi \) must satisfy the inequality constraint

\[
\frac{1}{1 - \frac{1}{\gamma} |(M_0 B_1)_{(i,:)}|} (M_0)_{(i,:)} \xi \leq 1.
\]

If \( \gamma > \gamma_{\text{opt}} \), then \( K_{j+\frac{1}{2}} = \text{Set}(M_{j+\frac{1}{2}}) \), where

\[
(M_{j+\frac{1}{2}})_{(i,:)} = \frac{1}{1 - \frac{1}{\gamma} |(M_0 B_1)_{(i,:)}|} (M_0)_{(i,:)}.
\]

We now construct \( M_j \) so that \( K_1 = \text{Set}(M_1) \). Recall that in this case

\[
K_1 = \{ x \in K_0 : Ax + B_2 u \in K_{\frac{1}{2}} \text{ for some } u \in \mathcal{R} \}.
\]

In terms of \( M_{\frac{1}{2}} \)

\[
K_1 = \{ x \in K_0 : M_{\frac{1}{2}} A x + M_{\frac{1}{2}} B_2 u \leq 1 \text{ for some } u \in \mathcal{R} \}.
\]

Thus, \( K_1 = \text{Set}(M_1) \), where \( M_1 = \left( \begin{array}{c} M_{\frac{1}{2}} \\ N \end{array} \right) \) for any

\[
N \in \text{Rack}[M_{\frac{1}{2}} A, M_{\frac{1}{2}} B_2].
\]

Given \( K_1 = \text{Set}(M_1) \), we may repeat the above process recursively as follows. First, generate if possible \( M_{j+\frac{1}{2}} \) based on \( M_j \). If not possible, then \( \gamma \leq \gamma_{\text{opt}} \). In general, we always can generate \( M_{j+1} \) based on \( M_{j+\frac{1}{2}} \).

This discussion is summarized in the following algorithm.

**Algorithm 5.1 (D = 0 Scalar Control):**

1) Initialize

\[
M_0 = \begin{pmatrix} C_1 \\ -C_1 \end{pmatrix}
\]

2) For each row of \( M_j \)
   a) If \( \frac{1}{\gamma} |(M_j B_1)_{(i,:)}| \geq 1 \), increase \( \gamma \) and restart.
   b) Otherwise, set

\[
(M_{j+\frac{1}{2}})_{(i,:)} = \frac{1}{1 - \frac{1}{\gamma} |(M_j B_1)_{(i,:)}|} (M_j)_{(i,:)}.
\]

3) Set

\[
M_{j+1} = \begin{pmatrix} M_j \\ N \end{pmatrix}
\]

for any

\[
N \in \text{Rack}[M_{\frac{1}{2}} A, M_{\frac{1}{2}} B_2].
\]

The algorithm will restart whenever \( \gamma \leq \gamma_{\text{opt}} \). Otherwise

\[
\text{CINV}(\text{dom}(U_\gamma)) = \bigcap_{j=0}^{\infty} \text{Set}(M_j).
\]

Note that the construction of the \( M_j \) does not require the solution of linear programs. However, linear programs are required for computationally efficient implementation to remove redundant constraints in the matrix descriptions of the various sets.
\( D_{11}, D_{12} \neq 0 \): Suppose now that \( D_{11} \neq 0 \) or \( D_{12} \neq 0 \).

Define
\[
V = \left\{ \zeta \in \mathbb{R}^p : \left| \zeta + \frac{1}{\gamma} D_{11} w \right| \leq 1, \ \forall |w| \leq 1 \right\}.
\]
Assuming \( V \) is nonempty, let \( M_{-\frac{1}{2}} \) be such that
\[
V = \text{Set}(M_{-\frac{1}{2}}).
\]

Then we can write
\[
U_{\gamma}(x) = \{ u \in \mathbb{R} : M_{-\frac{1}{2}}(C_1 x + D_{12} u) \leq 1 \}
\]
and
\[
\text{dom}(U_{\gamma}) = \{ x \in \mathbb{R}^n : M_{-\frac{1}{2}}(C_1 x + D_{12} u) \leq 1 \text{ for some } u \in \mathbb{R} \}.
\]
We see that \( \text{dom}(U_{\gamma}) = \text{Set}(M_0) \) for any
\[
M_0 \in \text{Rack}[M_{-\frac{1}{2}} C_1 \ M_{-\frac{1}{2}} D_{12}].
\]

Let \( K_0 = \text{Set}(M_0) \). We may construct \( K_1 = \text{Set}(M_1) \) based on \( K_0 \) as in Step 2) of the \( D = 0 \) Scalar Control algorithm. To construct \( M_1 \) so that \( K_1 = \text{Set}(M_1) \), recall that
\[
K_1 = \{ x \in K_0 : Ax + B_2 u \in K_1 \text{ for some } u \in U_{\gamma}(x) \}.
\]
Thus \( x \in K_1 \) only if there exists a \( u \) such that
\[
\left( \begin{array}{c}
M_{-\frac{1}{2}} C_1 \\
M_{-\frac{1}{2}} A
\end{array} \right) \left( \begin{array}{c}
\frac{x}{u}
\end{array} \right) \leq 1.
\]

Thus \( K_1 = \text{Set}(M_{-\frac{1}{2}}) \) for any
\[
N \in \text{Rack} \left[ \left( \begin{array}{cc}
M_{-\frac{1}{2}} C_1 & M_{-\frac{1}{2}} D_{12}
\end{array} \right) \right].
\]

We may repeat the above process recursively as follows.

\textbf{Algorithm 5.2 (}D \neq 0\text{ Scalar Control):}

1) Specify \( \gamma > 0 \).
   a) If any row of \( D_{11} \) satisfies
   \[
   \frac{1}{\gamma} |(D_{11})_{(i,:)}| \geq 1
   \]
   increase \( \gamma \) and restart.
   b) Otherwise ,
      i) Initialize
      \[
      M_{-\frac{1}{2}} = \left( \begin{array}{cc}
      \frac{1}{\gamma} |(D_{11})_{(i,:)}| & \frac{1}{\gamma} |(D_{11})_{(i,:)}|
      \end{array} \right).
      \]
      ii) Initialize
      \[
      M_0 \in \text{Rack}[M_{-\frac{1}{2}} C_1 \ M_{-\frac{1}{2}} D_{12}].
      \]

2) For each row of \( M_j \)
   a) If \( \frac{1}{\gamma} |(M_j B_1)_{(i,:)}| \geq 1 \), increase \( \gamma \) and restart.
   b) Otherwise, set
   \[
   (M_{j+\frac{1}{2}})_{(i,:)} = \frac{1}{1 - \frac{1}{\gamma} |(M_j B_1)_{(i,:)}|} (M_j)_{(i,:)}. \]

3) Set
   \[
   M_{j+1} = \left( \begin{array}{c}
   M_{j+\frac{1}{2}} \\
   N
   \end{array} \right)
   \]
   for any
   \[
   N \in \text{Rack} \left[ \left( \begin{array}{cc}
   M_{-\frac{1}{2}} C_1 & M_{-\frac{1}{2}} D_{12}
   \end{array} \right) \right].
   \]

Return to Step 2).

The algorithm will restart whenever \( \gamma \leq \gamma_{\text{opt}} \). Otherwise
\[
\text{CINV} \left( \text{dom}(U_{\gamma}) \right) = \bigcap_{j=0}^{\infty} \text{Set}(M_j).
\]

\textbf{D. Algorithm Implementation: Multivariable Control}

The multivariable control case requires only a slight modification of the scalar control algorithm.

\textbf{Definition 5.1:} For \( k > 1 \), define recursively \( \text{Rack}^k[M] \) as the set of matrices, \( M \), such that \( M \in \text{Rack}[M_{\text{temp}}] \) for some \( M_{\text{temp}} \in \text{Rack}^{k-1}[M] \).

An immediate consequence of this definition is the following.

\textbf{Proposition 5.2:} Let \( M \in \mathbb{R}^{(k_1 \times k_2)} \). Let \( \hat{M} \in \text{Rack}^k[M] \). Then
\[
\left( \begin{array}{c}
v \\
w
\end{array} \right) \in \text{Set}(M) \subset \mathbb{R}^{k_1 \times k_2}, \quad \text{for some } w \in \mathbb{R}^{k_2}.
\]

The only difference between the scalar and multivariable control cases is that successive applications of the Rack operator are required to express a combined constraint in the state and control as a constraint on the state only. For example, consider Step 1-b-ii) in the \( D \neq 0 \) Scalar Control algorithm. We wish to find an \( M_0 \) such that \( K_0 = \text{Set}(M_0) \), where
\[
K_0 = \{ x : M_{-\frac{1}{2}} C_1 x + M_{-\frac{1}{2}} D_{12} u \leq 1, \text{ for some } u \in \mathbb{R}^m \}.
\]

Thus any
\[
M_0 \in \text{Rack}^m[M_{-\frac{1}{2}} C_1 \ M_{-\frac{1}{2}} D_{12}]
\]
will suffice.

This leads to the following.

\textbf{Algorithm 5.3 (Multivariable Control):} Replace Rack in the \( D \neq 0 \) Scalar Control algorithm by \( \text{Rack}^m \), where \( m \) is the number of control variables.

The algorithm will restart whenever \( \gamma \leq \gamma_{\text{opt}} \). Otherwise
\[
\text{CINV} \left( \text{dom}(U_{\gamma}) \right) = \bigcap_{j=0}^{\infty} \text{Set}(M_j).
\]
E. Algorithm Termination and Control Law Construction

The algorithms presented thus far may be used to successively approximate CINV(dom(U)). However, these algorithms will terminate only when \( \gamma \leq \gamma_{opt} \). We still need a mechanism which guarantees an \textit{a priori} termination of the algorithm whenever a specified level of performance is achievable.

The following theorems address this issue.

\textbf{Theorem 5.2:} Consider the Controlled Invariance Kernel algorithm with \( K_0 = \text{dom}(U) \). Let \( \delta > 0 \). Define

\[
\bar{U}_\gamma(x) = \left\{ u \in \mathbb{R}^m : \left| C_1 x + \frac{1}{\gamma} D_{11} w + D_{12} u \right| \leq \sqrt{1 + \delta}, \ \forall |w| \leq 1 \right\}
\]

\[
\bar{K}_{j+\bar{\gamma}} = \left\{ \xi \in \mathbb{R}^n : \left| \xi + \frac{1}{\gamma} B_1 w \right| \leq 1 \right\}
\]

and

\[
\hat{K}_{j+1} = \left\{ x : Ax + B_2 u \in \bar{K}_{j+1} \text{ for some } u \in \bar{U}_\gamma(x) \right\}.
\]

If CINV(dom(U)) is nonempty, then

\[
K_j \subset \hat{K}_{j+1}
\]

for sufficiently large \( j \).

Furthermore, \( K_j \) is controlled invariant under the controlled difference inclusion defined by \((F_{\gamma^{1+\delta}}, \bar{U}_{\gamma}, f)\) in (3).

\textbf{Proof:} We first show that \( K_{j+\bar{\gamma}} \) is strictly contained in \( \bar{K}_{j+\bar{\gamma}} \).

Suppose \( \xi \in \bar{K}_{j+\bar{\gamma}} \). Then

\[
\xi + \frac{1}{\gamma} B_1 w \in K_j, \ \forall |w| \leq 1.
\]

Similarly

\[
\xi + \frac{1}{\gamma} B_1 w \in K_j, \ \forall |w| \leq \sqrt{1 + \delta}.
\]

Therefore

\[
\xi + \frac{1}{\gamma} B_1 w \in K_j, \ \forall |w| \leq \sqrt{1 + \delta} - 1.
\]

This shows that \( \xi \in \bar{K}_{j+\bar{\gamma}} \) implies

\[
\xi + \frac{1}{\gamma} B_1 w \in \hat{K}_{j+1}, \ \forall |w| \leq \sqrt{1 + \delta} - 1.
\]

Since \( B_1 \) has rank \( n \), \( \hat{K}_{j+1} \) is strictly contained in \( \bar{K}_{j+1} \).

Now suppose \( x \in \hat{K}_{j+1} \). Then \( Ax + B_2 u \in \bar{K}_{j+1} \) for some \( u \in U_\gamma(x) \). Since \( \bar{K}_{j+1} \) is strictly contained in \( \bar{K}_{j+1} \) and since \( \bar{K}_{j+1} \) is compact, there exists an \( \alpha > 0 \) such that for all \( |v| \leq \alpha \)

\[
A(x + v) + B_2 u \in \bar{K}_{j+1}
\]

and

\[
|C_1 v| \leq \sqrt{1 + \delta} - 1.
\]

The latter inequality implies \( u \in \bar{U}_{\gamma}(x + v) \). These arguments show that \( x \in \hat{K}_{j+1} \) implies

\[
x + v \in \bar{K}_{j+1}, \ \forall |v| \leq \alpha.
\]

It is important to note that \( \alpha \) is independent of \( j \).

With \( \alpha \) as above, if CINV(dom(U)) is nonempty, then for sufficiently large \( j \), \( K_j \), and \( \hat{K}_{j+1} \) are both close to CINV(dom(U)). More precisely, \( x \in K_j \) implies \( x + v \in \hat{K}_{j+1} \) for some \( |v| \leq \alpha/2 \). This in turn implies \( x \in \hat{K}_{j+1} \) which proves the desired containment. That \( K_j \) is controlled invariant under the controlled difference inclusion \((F_{\gamma^{1+\delta}}, \bar{U}_{\gamma}, f)\) follows immediately.

\textbf{Theorem 5.3:} In the framework of Theorem 5.2, let \( K_j = \text{Set}(M_j) \) and \( \hat{K}_{j+1} = \text{Set}(\hat{M}_{j+1}) \). Define

\[
J(\rho) \overset{\text{def}}{=} \max_x \rho^T x
\]

subject to

\[
M_j x \leq 1.
\]

Then

\[
K_j \subset \hat{K}_{j+1}
\]

if and only if \( J(\rho) \leq 1 \) for all rows \( \rho^T = (\hat{M}_{j+1})_{(i,:)} \).

\textbf{Proof:} The proof is clear from the definition of \( J(\rho) \).

\textbf{Theorem 5.4:} In the framework of Theorem 5.2, suppose

\[
K_j \subset \hat{K}_{j+1}.
\]

There exists an internally stabilizing \( K_{st} \) which achieves a performance of \( \gamma(1 + \delta) \).

\textbf{Proof:} The scalar and multivariable control cases are treated separately.

Let \( \hat{K}_{j+1} = \text{Set}(\hat{M}_{j+1}) \). Define

\[
\hat{V} = \left\{ \xi \in \mathbb{R}^n : \left| \xi + \frac{1}{\gamma + \delta} D_{11} w \right| \leq 1, \ \forall |w| \leq 1 \right\}.
\]

Let \( \hat{M}_{\gamma} \) be such that \( \hat{V} = \text{Set}(\hat{M}_{\gamma}) \). Let

\[
M^* = \begin{pmatrix}
M_{\gamma} C_1 + \frac{1}{\gamma + \delta} M_{\gamma} D_{12} \sqrt{1 + \delta} \\
M_{\gamma} A \end{pmatrix}
\]

Define \( \tilde{g} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by

\[
\tilde{g}(x) = \phi^\ast (x)
\]

where \( \phi^\ast \) is defined in Proposition 2.2. By construction, if \( x(t) \in K_j \), then

\[
Ax(t) + \frac{1}{\gamma + \delta} B_1 w + B_2 \tilde{g}(x(t)) \in K_j, \ \forall |w| \leq 1.
\]

Furthermore

\[
|C_1 x(t) + \frac{1}{\gamma} D_{11} w + D_{12} \tilde{g}(x(t))| \leq \sqrt{1 + \delta}, \ \forall |w| \leq 1.
\]

As in the proof of Theorem 5.1, define \( p : \mathbb{R}^n \rightarrow \mathbb{R}^+ \) by

\[
p(x) = \inf \left\{ \alpha \in \mathbb{R}^+ : x \in \alpha K_j \right\}.
\]
Define $g: \mathcal{R}^n \rightarrow \mathcal{R}$ by

$$g(x) = p(x)\hat{g}(x/p(x)).$$

The arguments in [34] show that this feedback is internally stabilizing with a performance of $\gamma(1 + \delta)$. Note that we also could have used $\phi(M)$ to define $\hat{g}$ or even any convex combination of the two.

The multivariable case is similar. For notational simplicity, we consider the case where $m = 2$. With $M^*$ defined as above, pick $M^* \in \text{Rack}[M^*]$. Define $\hat{g}_1: \mathcal{R}^n \rightarrow \mathcal{R}$ by

$$\hat{g}_1(x) = \phi^M^*(x).$$

Define $\hat{g}_2: \mathcal{R}^n \rightarrow \mathcal{R}$ by

$$\hat{g}_2(x) = \phi^M^*\begin{pmatrix} x \\ \hat{g}_1(x) \end{pmatrix}.$$

Set

$$\hat{g}(x) = \begin{pmatrix} \hat{g}_1(x) \\ \hat{g}_2(x) \end{pmatrix}.$$

The remainder of the proof follows the scalar control case.

The main idea here is that the multivariable control case requires successive applications of the Rack operator. In defining a control law, we simply unwrap these Rack applications to first define $\hat{g}_1$, and then $\hat{g}_2$ based on $\hat{g}_1$. For $m = 3$, $\hat{g}_3$ would be based on both $\hat{g}_1$ and $\hat{g}_2$ and so on. We then may apply the same scaling argument to complete the proof. \qed

Note that the proof of Theorem 5.4 provides an explicit construction of the desired state feedback. While the state feedback is continuous, it is not differentiable in general. If it were differentiable, then arguments in [18] show that linear static feedback can deliver the same performance.

Theorems 5.2–5.4 together suggest an easily implementable method to terminate any of the control algorithms for a specified desired performance as follows:

- Express the desired performance as $\gamma(1 + \delta)$.
- Initialize $K_0 = \text{dom}(U_\gamma)$.
- If $\gamma \leq \gamma_{opt}$, the algorithms will terminate and $\gamma$ must be increased.
- If $\gamma > \gamma_{opt}$, use Theorem 5.3 to terminate the algorithms whenever $K_j \subset K_{j+1}$. The continuous control law from Theorem 5.4 is internally stabilizing with a performance of $\gamma(1 + \delta)$, and $\gamma$ may be decreased if desired.

Finally, care should be taken in passing from $M_j$ to $M_{j+1}$ to add only rows which introduce new constraints. Once $M_{j+1}$ is generated, newly redundant constraints corresponding to old rows from $M_j$ should be removed as well.

VI. EXAMPLES

The following example was considered in [34] where linear dynamic feedback was used to derive a nonlinear static feedback.

**Example 6.1:** Consider the system

$$x(t + 1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

$$z(t) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x(t).$$

We will apply the $D = 0$ Scalar Control algorithm for $\gamma = 3$. We first initialize

$$\text{dom}(U_\gamma) = \text{Set}(M_0)$$

where

$$M_0 = \begin{pmatrix} C_1 \\ -C_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Following the algorithm

$$(M_{1\gamma})_{(i,:)} = \frac{1}{1 - \frac{1}{2} [(M_0 B_1)_{(i,:)}] (M_0)_{(i,:)}}.$$

In this case

$$M_{1\gamma} = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \\ -3/2 & 0 \\ 0 & -3/2 \end{pmatrix}.$$

Using the Rack notation

$$M_1 = \begin{pmatrix} M_0 \\ N \end{pmatrix}$$

for any

$$N \in \text{Rack}[[M_{1\gamma} A \quad M_{1\gamma} B_2]].$$

In this case

$$(M_{1\gamma} A \quad M_{1\gamma} B_2) = \begin{pmatrix} 0 & 3/2 & 0 \\ 3/2 & 0 & 3/2 \\ -3/2 & 0 & -3/2 \end{pmatrix}.$$}

Following the procedure in Proposition 2.1, we separate the rows into three groups depending on the sign of the element in the last column. In this case

$$Z_+ = \{2\}, \quad Z_- = \{4\}, \quad Z_0 = \{1, 3\}.$$

Computing the single pairwise comparison row leads to

$$\rho^2_{24} = \begin{pmatrix} 0 & 0 \end{pmatrix},$$

which imposes no constraint. The remaining rows lead to

$$N = \begin{pmatrix} 0 & 3/2 \\ 0 & -3/2 \end{pmatrix}$$

and

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 0 & 3/2 \\ 0 & -3/2 \end{pmatrix}.$$
We see that some of the constraints represented by the rows in $M_1$ are redundant. A minimal representation is

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3/2 \\ -1 & 0 \\ 0 & -3/2 \end{pmatrix}.$$  

Repeating the entire procedure leads to $M_2 = M_1$.

The previous argument shows that $\text{Set}(M_1)$ is controlled invariant provided $\|w\| \leq 1/3$. To construct a feedback law following the proof of Theorem 5.4, let

$$M^* = \begin{pmatrix} M_1 A & M_1 B_2 \\ 0 & 3/2 & 0 \\ 0 & -3/2 & 0 \end{pmatrix}.$$  

Then the feedback

$$\tilde{g}(x) = \phi^M_0 (x) = 1/3 - x_1,$$

makes $\text{Set}(M_1)$ invariant for $\|w\| \leq 1/3$. Let $p(\cdot)$ be the norm defined by $\text{Set}(M_1)$. Then

$$p(x) = \max \left\{ \|x_1\|, \frac{3}{2} \|x_2\| \right\}.$$  

This leads to the scaled global feedback law

$$\tilde{g}(x) = p(x) \phi^M_0 (x) = p(x) (1/3 - x_1) / p(x) = \frac{1}{3} p(x) - x_1.$$  

This feedback is internally stabilizing with a performance of $\gamma = 3$.

We also could define $\tilde{g}$ with

$$\tilde{g}(x) = \phi^M_+ (x) = -1/3 - x_1.$$  

Another possibility is

$$\tilde{g}(x) = \left( \phi^M_0 (x) + \phi^M_+ (x) \right) / 2 = -x_1.$$  

This last selection leads to a scaled global feedback law which is linear.

The following example was considered in [19].

**Example 6.2:** Define

$$P_\alpha(\lambda) = \frac{\lambda (\kappa \lambda^2 - 2.5\lambda + 1)}{(1 - 0.2\lambda)(23\lambda^2 - 2.5\lambda + 1)}$$

where $\lambda$ denotes the unit delay operator and $\kappa$ is a parameter. The regulated output is given by

$$z = P_\alpha w + P_\alpha u.$$  

A state-space realization is

$$x(t+1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4.6 & -23.5 & 2.7 \end{pmatrix} x(t)$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} w(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(t).$$

$$z(t) = (\kappa - 2.5) x(t).$$

For $\kappa = 1.51$, [19] showed that the optimal linear controller is of order nine and $\gamma_{\text{opt}} \approx 3.07$.

To apply the $D = 0$ Scalar Control algorithm, $B_1$ and $C_1$ must be modified to satisfy the required rank assumptions. Toward this end, we modify $B_1$ and $C_1$ to

$$B_1 = \begin{pmatrix} 0 & 0 & -10^{-6} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1.51 & -2.5 & 1 \\ -0.5 & 0 & 0 \\ 0 & -0.5 & 0 \end{pmatrix}.$$  

The modification of $B_1$ reflects that the rank $\kappa$ assumption on $B_1$ conveniently assures the nonempty interior in Lemma 5.1, strict containment in Theorem 5.2, and exponential stabilization in Theorems 5.1 and 5.4. The modification of $C_1$ differs in that the rank $\kappa$ assumption does more than assure compactness of the desired controlled invariant set. The matrix $C_1$ is the only mechanism which bounds the state magnitude in closed-loop operation. Therefore, a modification of the same order as in $B_1$ would allow the states to have magnitudes of the
order $10^5$. While theoretically acceptable, such "infinitesimal" modifications of $C_1$ led to numerical problems where the $D = 0$ Scalar Control algorithm failed to terminate.

The termination procedure described by Theorems 5.2 and 5.3 was employed with $\delta = 0.01$. Successive iterations led to $\gamma = 3.07$. For these values, $K_{\text{final}} = \text{Set}(M_{\text{final}})$ is a controlled invariant set as in Theorem 5.2, and

$$M_{\text{final}} = \begin{pmatrix}
1.51 & -2.5 & 1 \\
-0.501 & 0 & 0 \\
0 & -0.501 & 0 \\
0 & 0 & -0.501 \\
0 & -1.171 & 1.4712 \\
0 & -1.5472 & 1.5534 \\
0 & -1.5788 & 1.6046 \\
0 & -1.5985 & 1.6559 \\
0 & -1.6591 & 1.6878 \\
0 & -1.6661 & 1.6540 \\
-1.51 & 2.5 & -1 \\
\vdots \\
0 & 0.6436 & -1.0655
\end{pmatrix}$$

We see that $K_{\text{final}}$ is described by 22 inequality constraints. Because of the symmetry with respect to the origin, the bottom 11 rows are the negatives of the top 11 rows.

The computation of $M_{\text{final}}$ required linear programs to remove redundant constraints and to test the stopping criterion in Theorem 5.3. In the dual formulation, these linear programs had three constraints and up to 22 variables. However, several (approximately 5000) such linear programs were solved, mostly due to redundancy in the Rack operator. Larger problems could benefit from more efficient methods of removing redundant constraints.

**Example 6.3**: We now modify the previous example to incorporate a penalty on the controls. The new regulated output is

$$z = \begin{pmatrix} P_r w + P_u u \\ \alpha u \end{pmatrix}.$$  

For $\kappa = 2$ and $\alpha = 0.1$, [19] showed that the optimal linear controller is second order and $\gamma_{opt} \approx 4.27$.

To apply the $D \neq 0$ Scalar Control algorithm, $B_1$ and $C_1$ were modified as before to satisfy the rank $n$ assumption. The termination procedure described by Theorems 5.2 and 5.3 was employed with $\delta = 0.01$. Successive iterations led to $\gamma = 4.29$.

For these values, $K_{\text{final}} = \text{Set}(M_{\text{final}})$ is a controlled invariant set as in Theorem 5.2, and $K_{\text{final}}$ is described by 18 inequality constraints.

The control law derived from $(\phi^M_0 + \phi^M_0)/2$ as described in the proof of Theorem 5.4 was constructed. Fig. 1 shows the response of $x_1(i)$ to an initial condition of $z(0) = (1 \ -2 \ 3)^T$. Fig. 2 shows the regulated output norm response to a random disturbance of magnitude $\pm 1$ up to time 100 and $\pm 2$ thereafter. The regulated output satisfies the expected bounds.

**VII. CONCLUDING REMARKS**

This paper has derived a constructive approach to deriving static near-optimal nonlinear controllers for $\ell^\infty$-induced norm minimization. The central idea was the relation between disturbance rejection and controlled invariance of a particular subset of the state space.

Although we explicitly construct the desired nonlinear feedback, it is unclear what computational advantage—if any—these methods have over existing computations of linear controllers as in [20]. Furthermore, even though the nonlinear feedback is static, the complexity of the controller is comparable to or even greater than linear dynamic feedback.

The nonlinear feedback complexity grows as the number of constraints to describe the desired controlled invariant set grows. Computational experience indicates that this number is proportional to the order of the required linear dynamic feedback.

The present methods do not extend readily to noisy measurements or output feedback. A conjecture is that a separation structure could follow from the construction of a "set-valued" observer, i.e., an observer which produces a set of possible
state-feedback control values based on observed measurements.

REFERENCES


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