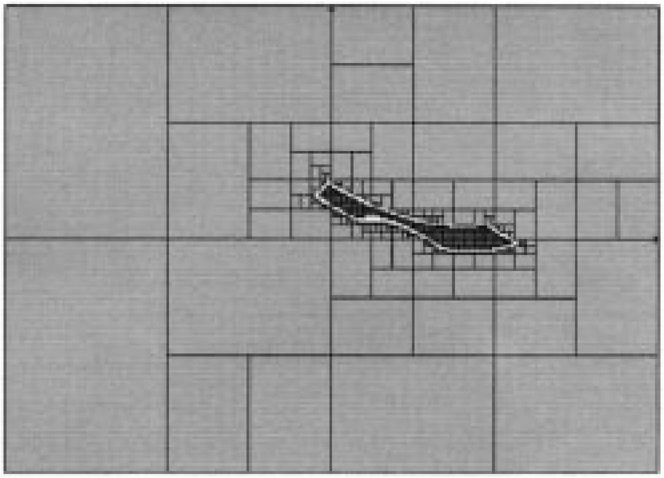

Fig. 2. Paving generated by SIVIA for Example 3. The frame is as in Fig. 1.

respectively generate the state sequences

$$\mathbf{x}_a(0) = (1, 2)^T$$
$$\mathbf{x}_a(1) = (-0.3892, 2.946)^T$$
$$\mathbf{x}_a(2) = (-0.000\,03, 0.0001)^T$$

and (21)

$$\mathbf{x}_b(0) = (1, 2)^T$$
$$\mathbf{x}_b(1) = (-0.000\,05, 2.2606)^T$$
$$\mathbf{x}_b(2) = (-0.000\,05, 0.000\,00)^T.$$

Example 3: Consider the same system, but assume now that we only want to drive it in two steps into the box $[-0.12, 0.12]^2$. This amounts to characterizing the set

$$\mathbb{V} = \mathbf{g}^{-1}([-0.12, 0.12]^2). \tag{22}$$

For $\varepsilon_r = 0.01$ and the same prior domain of interest for $\mathbf{v}$ as in Example 2, in 8 s, SIVIA brackets $\mathbb{V}$ between two subpavings as illustrated by Fig. 2. Boxes in dark grey belong to $\mathbb{V}^-$ and have thus been proved to belong to $\mathbb{V}$. Those in light grey have been eliminated. The uncertainty layer is in white. The complexity of the problem increases exponentially with the dimension of the accumulation set of the paving [7], which is one in this example instead of zero in Example 2. This explains why the computing time is larger than in Example 2, although ε_r is larger. Any $\mathbf{v} \in \mathbb{V}^-$ is guaranteed to send the state into $\mathbb{X}_t$.

Note that if $\mathbb{X}_t = [-0.05, 0.05]^2$, for the same required accuracy ε_r and prior domain of interest for the control, SIVIA numerically proves the nonconnexity of $\mathbb{V}$.

V. Conclusion

By taking advantage of the guaranteed nature of the numerical results provided by interval analysis, it is possible to solve the problem of computing all sequences of controls driving a deterministic nonlinear discrete-time state-space system from a given initial state to a given desired set of terminal states. To the best of our knowledge, no other guaranteed method is available for that purpose. Taking additional inequality constraints on the state or input into account would be particularly simple.

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Linear Nonquadratic Optimal Control

Jeff S. Shamma and Dapeng Xiong

***Abstract*—We consider the optimization of nonquadratic measures of the transient response. We present a computational implementation of dynamic programming recursions to solve finite-horizon problems. In the limit, the finite-horizon performance converges to the infinite-horizon performance. We provide conditions based on finite-horizon computations which only assure that a receding horizon implementation of the finite-horizon optimal control is stabilizing and within a specified tolerance of the infinite-horizon performance.**

***Index Terms*—Dynamic programming, ℓ-optimal control, receding horizon.**

I. Introduction

A popular design paradigm for linear time-invariant systems is linear-quadratic (LQ) optimal control [6]. Given a discrete-time linear

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system of the form

$$x(k+1) = Ax(k) + Bu(k), \qquad x(0) = x_o \tag{1}$$

the LQ cost function is

$$J_{LQ}(x_o) = \inf_{u(\cdot)} \sum_{k=0}^{\infty} x^T(k)Qx(k) + u^T(k)Ru(k)$$

where Q and R are design parameters.

In this paper, we consider cost functions of the form

$$J(x_o) = \inf_{u(\cdot)} \sum_{k=0}^{\infty} h(x(k), u(k))$$

where the nonquadratic penalty function, $h(\cdot,\cdot)$, defines a norm on the state and control vector. For example, given vectors $x \in \mathcal{R}^n$ and $u \in \mathcal{R}^m$, we may define

$$h(x,u) = \max\{|x_1|, \cdots, |x_n|, |u_1|, \cdots, |u_m|\}$$

or alternatively

$$h(x,u) = \sum_{i=0}^{n} |x_i| + \sum_{j=0}^{m} |u_j|.$$

In words, the above performance index essentially reflects the ℓ^1-norm of the transient response. Since $h(\cdot,\cdot)$ defines a norm and has linear growth, it does not correspond to any quadratic function.

First, we present a computational implementation of dynamic programming recursions to solve the associated finite-horizon problem

$$J_N(x_o) = \inf_{u(\cdot)} \sum_{k=0}^{N} h(x(k), u(k)).$$

This procedure leads to a set of matrices, M_N, so that the optimal finite-horizon cost is given by

$$\max_i (M_N x_o)_i.$$

We then consider a receding horizon implementation of the finite-horizon optimal control. Receding horizon control, also known as model predictive control, has been considered by several authors for linear and nonlinear systems (cf., [5], [7], [8], [9], [10], [11], [14] and references therein). Many of these references exploit the receding horizon formulation in order to address state and control constraints. In this paper, we consider only the unconstrained case.

Reference [5] has shown in a fairly general setting that for a sufficiently long horizon length, the receding horizon control law is stabilizing. Furthermore, the receding horizon control performance approaches the infinite-horizon performance as the horizon length increases. In this paper, we provide computational criteria which assure that a receding horizon implementation of the finite-horizon optimal control is both stabilizing and within a specified tolerance of the infinite-horizon performance. While these criteria do not provide an *a priori* bound on the required horizon length, they are based on finite-horizon computations only.

A common practice in receding horizon control is to impose a terminal condition on the finite-horizon problems (such as $x(N+1) = 0$) in order to assure that the receding horizon control is stabilizing. In this paper, we do not impose such a terminal constraint.

The resulting near-optimal controllers are piecewise linear functions of the state. Although nonlinear and not necessary unique, the receding horizon control law can be made to be globally Lipshitz continuous. The utility of nonlinear controllers for linear unconstrained systems has been considered for certain disturbance rejection problems [1], [3], [12], [13].

II. Preliminaries

Let $\mathcal{Z}^+$ denote the set of nonnegative integers. For $x \in \mathcal{R}^n$, let x_i denote the ith component of x. Let $|\cdot|_p$ denote the usual p-norms on $\mathcal{R}^n$.

Let $S \subset \mathcal{R}^n$ be a convex compact set of nonempty interior containing the origin. Assume further that S is symmetric with respect to the origin, i.e.,

$$x \in S \Leftrightarrow -x \in S.$$

We will call such sets Minkowski sets. The Minkowski functional [2, p. 106], $p(\cdot; S): \mathcal{R}^n \rightarrow \mathcal{R}^+$, associated with S is defined as

$$p(x; S) = \inf\{\alpha \in \mathcal{R}^+ : x \in \alpha S\}.$$

Under the stated assumptions on S, $p(\cdot; S)$ defines a norm on $\mathcal{R}^n$.

Let $\mathbf{1}$ denote a column vector of appropriate length with unit elements. For $M \in \mathcal{R}^{z\times n}$, let $\mathrm{Set}(M)$ denote the subset of $\mathcal{R}^n$ associated with M defined by the constraints

$$\mathrm{Set}(M) = \{x : Mx \le \mathbf{1}\}.$$

For $M_1 \in \mathcal{R}^{z\times n}$, $M_2 \in \mathcal{R}^z$ consider the subset, S, of $\mathcal{R}^n$ defined by

$$S = \{x : M_1 x + M_2 w \le \mathbf{1} \text{ for some } w \in \mathcal{R}\}.$$

Define

$$\mathrm{Rack}[(M_1 \quad M_2)] = \{\tilde{M} \in \mathcal{R}^{\tilde{z}\times n} : S = \mathrm{Set}(\tilde{M})\}$$

i.e., $\mathrm{Rack}[(M_1 \quad M_2)]$ is the set of matrices which give a direct characterization of S. While the set, S, is unique, its matrix representation is not. Hence, $\mathrm{Rack}[(M_1 \quad M_2)]$ represents a *set* of possible matrix representations. Nevertheless, by abuse of notation, we will write $\tilde{M} = \mathrm{Rack}[(M_1 \quad M_2)]$ intending that $\tilde{M}$ is any selection. The construction of an element of $\mathrm{Rack}[(M_1 \quad M_2)]$ may be achieved through the Fourier–Motzkin algorithm which is described in [4] and reviewed in the Appendix.

Proposition 2.1: Let $\tilde{M} = \mathrm{Rack}\big[\big(M_1 \quad M_2\big)\big]$. Then

$$\min_u \max_i (M_1 x + M_2 u)_i = \max_j (\tilde{M}x)_j.$$

Proof: For $\alpha \neq 0$

$$\begin{aligned} M_1 x + M_2 u \le \alpha\mathbf{1} &\Leftrightarrow M_1 \frac{1}{\alpha} x + M_2 \frac{1}{\alpha} u \\ \le \mathbf{1} &\Leftrightarrow \tilde{M}\frac{1}{\alpha} x \le \mathbf{1} \Leftrightarrow \tilde{M}x \le \alpha\mathbf{1} \end{aligned}$$

which implies the desired result. □

For $M_1 \in \mathcal{R}^{z_1\times n}$ and $M_2 \in \mathcal{R}^{z_2\times n}$, consider the subset

$$S = \{x : \max_i (M_1 x)_i + \max_j (M_2 x)_j \le 1\}.$$

Define

$$\mathrm{Pair}[M_1, M_2] = \{M_{12} \in \mathcal{R}^{z_{12}\times n} : S = \mathrm{Set}(M_{12})\}$$

i.e., $\mathrm{Pair}[M_1, M_2]$ is the set of matrices which give a direct characterization of S. The matrix M_{12} can be constructed by forming a pairwise combination of the rows in M_1 and M_2 as follows.

Suppose

$$M_1 = \begin{pmatrix} v_1 \\ \vdots \\ v_{z_1} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} w_1 \\ \vdots \\ w_{z_2} \end{pmatrix}.$$

Then

$$M_{12} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_1 + w_{z_2} \\ \vdots \\ v_{z_1} + w_1 \\ \vdots \\ v_{z_1} + w_{z_2} \end{pmatrix}.$$

This leads to $\mathrm{Set}(M_{12})$ being described by $z_1 \times z_2$ inequalities. Appropriate linear programs then can be solved to remove redundant inequalities. Again, by abuse of notation, we will write $M_{12} = \mathrm{Pair}[M_1, M_2]$.

III. Main Results

We will consider the discrete-time linear system (1) with $x(k) \in \mathcal{R}^n$ and $u(k) \in \mathcal{R}$ (extensions to the multi-input case require only notational changes).

For $\Gamma_X \in \mathcal{R}^{z\times n}$ and $\Gamma_U \in \mathcal{R}^z$, we define the following objective function:

$$J(x_o) = \inf_{u(\cdot)} \sum_{k=0}^{\infty} h(x(k), u(k))$$

where

$$h(x(k), u(k)) = \max_i (\Gamma_X x(k) + \Gamma_U u(k))_i.$$

As mentioned in Section I, such a form can represent a variety of piecewise-linear penalty functions.

We make the following assumptions throughout.

Assumption 3.1:

1) $[A, B]$ is a stabilizable pair.
2) $\mathrm{Set}((\Gamma_X \quad \Gamma_U))$ is a Minkowski set in $\mathcal{R}^{n+1}$.

Related to $J(x_o)$ is the finite-horizon optimization

$$J_N(x_o) = \min_{u(\cdot)} \sum_{k=0}^{N} h(x(k), u(k))$$

where $N \in \mathcal{Z}^+$.

In the discussion that follows, we will show that the infinite-horizon optimal performance can be approached by a receding horizon implementation of the finite-horizon optimal policy (cf., [4]). Furthermore, we will derive criteria based on only *finite-horizon* computations which guarantee that the infinite-horizon optimal performance is achieved within a specified level of accuracy.

We begin by characterizing finite-horizon optimal performance. For $N \in \mathcal{Z}^+$, define

$$K_N = \{x_o : J_N(x_o) \le 1\}.$$

Theorem 3.1: The set $K_0 = \mathrm{Set}(M_0)$, where

$$M_0 = \mathrm{Rack}[(\Gamma_X \quad \Gamma_U)].$$

Furthermore, $K_N = \mathrm{Set}(M_N)$, where

$$M_N = \mathrm{Rack}[\mathrm{Pair}[(\Gamma_X \quad \Gamma_U),\ (M_{N-1}A \quad M_{N-1}B)]].$$

Proof: Clearly

$$x_o \in K_0 \Leftrightarrow x_o \in \mathrm{Set}(M_0).$$

Furthermore, by Proposition 2.1

$$J_0(x_o) = \max_i (M_0 x_o)_i.$$

By a standard dynamic programming argument, $x_o \in K_1$ if and only if there exists a $v \in \mathcal{R}$ such that

$$h(x_o, v) + J_0(Ax_o + Bv) \le 1.$$

Alternatively

$$\begin{aligned} x_o \in K_1 &\Leftrightarrow \min_{v\in\mathcal{R}} (\max_i (\Gamma_X x_o + \Gamma_U v)_i \\ &\qquad + \max_j (M_0 A x_o + M_0 B v)_j) \le 1 \\ &\Leftrightarrow \min_{v\in\mathcal{R}} \max_i \left(M_X^{\mathrm{temp}} x_o + M_U^{\mathrm{temp}} v\right)_i \le 1 \\ &\Leftrightarrow x_o \in \mathrm{Set}\left(\mathrm{Rack}\left[\left(M_X^{\mathrm{temp}} \quad M_U^{\mathrm{temp}}\right)\right]\right) \end{aligned}$$

for some

$$\left(M_X^{\mathrm{temp}} \quad M_U^{\mathrm{temp}}\right) = \mathrm{Pair}[(\Gamma_X \quad \Gamma_U),\ (M_0 A \quad M_0 B)].$$

Proceeding recursively completes the proof. □

The main idea of the proof is that via Proposition 2.1, the matrices M_N not only characterize K_N, but also provide optimal cost-to-go values. Note that since the penalty function $h(\cdot,\cdot)$ has linear growth

$$J_N(\cdot) = p(\cdot; K_N) \tag{2}$$

i.e., K_N not only describes a single level set of $J_N(\cdot)$, but completely characterizes $J_N(\cdot)$.

Proposition 3.1:

1) For any $N \in \mathcal{Z}^+$, K_N is a Minkowski set.
2) $K_\infty \stackrel{\mathrm{def}}{=} \bigcap_{N\in\mathcal{Z}^+} K_N$ is a Minkowski set.

Proof: The proposition is a direct consequence of Assumption 3.1. □

Proposition 3.2: Define

$$\alpha_N = \min\{\alpha : K_N \subset \alpha K_{N+1}\}.$$

Then

$$\lim_{N\to\infty} \alpha_N = 1.$$

Proof: Exploit the fact that the K_N form a nested sequence of compact sets. □

Note that

$$K_{N+1} \subset K_N \subset \alpha_N K_{N+1}$$

implies

$$J_{N+1}(x) \ge J_N(x) \ge \frac{1}{\alpha_N} J_{N+1}(x), \qquad \forall x \in \mathcal{R}^n.$$

It is important to note that the α_N are a result of *finite-horizon* computations.

We now define a receding horizon implementation of a finite-horizon policy. Given $N \in \mathcal{Z}^+$, define $\phi(\cdot; N) : \mathcal{R}^n \to \mathcal{R}$ as

$$\phi(x; N) = \arg\min_{v\in\mathcal{R}} \{h(x, v) + J_{N-1}(Ax + Bv)\}.$$

From Theorem 3.1 and (2), we see that $\phi(x; N)$ can be computed by an appropriate linear program.

The receding horizon policy repeatedly implements the first step of a finite-horizon policy. Key issues are then whether the receding horizon policy is stabilizing, and if so, what is the resulting infinite-horizon performance.

Since the K_N and K_∞ are Minkowski sets, they define norms on $\mathcal{R}^n$ which are equivalent to $\max|\cdot|_\infty$. We then can define the constants $\underline{\beta}_N$, $\bar{\beta}_N$, $\underline{\beta}_\infty$, and $\bar{\beta}_\infty$ so that for all $x \in \mathcal{R}^n$

$$\underline{\beta}_N |x|_\infty \le J_N(x) \le \bar{\beta}_N |x|_\infty$$
$$\underline{\beta}_\infty |x|_\infty \le p(x; K_\infty) \le \bar{\beta}_\infty |x|_\infty.$$

Because of the set containments $K_0 \supset K_1 \supset \cdots \supset K_\infty$, we may select these constants so that

$$\underline{\beta}_0 \le \underline{\beta}_N \le \bar{\beta}_N \le \bar{\beta}_\infty.$$

Note that

$$\underline{\beta}_0 |x|_\infty \le J_0(x) \le h(x,u)$$

for all $x \in \mathcal{R}^n$ and $u \in \mathcal{R}$.

Theorem 3.2: Let N^* be such that for all $N \ge N^*$

$$\underline{\beta}_0 - (\alpha_{N-1} - 1)\bar{\beta}_N > 0.$$

The receding horizon policy $\phi(\cdot; N)$ is stabilizing for all $N \ge N^*$. Furthermore, $J_N(\cdot)$ is a Lyapunov function for the closed-loop system so that

$$J_N(x(k+1)) \le \gamma_N J_N(x(k))$$

where

$$\gamma_N = \left(1 - \frac{\underline{\beta}_0 - (\alpha_{N-1} - 1)\bar{\beta}_N}{\bar{\beta}_N}\right).$$

Proof: Let $x(k)$ and $u(k)$ be the state and control trajectory, respectively, resulting from the receding horizon policy

$$u(k) = \phi(x(k); N).$$

For any $k \in \mathcal{Z}^+$

$$J_N(x(k)) = h(x(k), u(k)) + J_{N-1}(x(k+1)).$$

Therefore

$$\begin{aligned}
&J_N(x(k)) - J_N(x(k+1)) \\
&\quad = h(x(k), u(k)) + J_{N-1}(x(k+1)) - J_N(x(k+1)) \\
&\quad \ge h(x(k), u(k)) + J_{N-1}(x(k+1)) - \alpha_{N-1} J_{N-1}(x(k+1)) \\
&\quad \ge h(x(k), u(k)) - (\alpha_{N-1} - 1) J_N(x(k)) \\
&\quad \ge (\underline{\beta}_0 - (\alpha_{N-1} - 1)\bar{\beta}_N)|x(k)|_\infty \\
&\quad \ge \left(\frac{\underline{\beta}_0 - (\alpha_{N-1} - 1)\bar{\beta}_N}{\bar{\beta}_N}\right) J_N(x(k))
\end{aligned}$$

which completes the proof. □

It is important to note that the condition in Theorem 3.2 is based on finite-horizon computations only.

We now bound the performance achieved by a stabilizing receding horizon policy.

Theorem 3.3: Let N^* and γ_N be as in Theorem 3.2. Under the receding horizon policy

$$u(k) = \phi(x(k); N)$$

the infinite-horizon performance satisfies

$$\sum_{k=0}^{\infty} h(x(k), u(k)) \le \left(1 + \left(\frac{\alpha_{N-1} - 1}{\alpha_{N-1}}\right)\frac{\gamma_N}{1 - \gamma_N}\right) J_N(x(0))$$

for all $N \ge N_*$.

Proof: We can bound $h(x(k), u(k))$ term-by-term as follows. First

$$\begin{aligned}
&h(x(0), u(0)) \\
&\quad = J_N(x(0)) - J_{N-1}(x(1)) \\
&\quad = J_N(x(0)) - J_N(x(1)) + J_N(x(1)) - J_{N-1}(x(1)) \\
&\quad \le J_N(x(0)) - J_N(x(1)) + J_N(x(1)) - \frac{1}{\alpha_{N-1}} J_N(x(1)).
\end{aligned}$$

Similarly

$$h(x(1), u(1)) \le J_N(x(1)) - J_N(x(2)) + \left(\frac{\alpha_{N-1} - 1}{\alpha_{N-1}}\right) J_N(x(1)).$$

A summation of similar bounds on $h(x(k), u(k))$ leads to

$$\begin{aligned}
\sum_{k=0}^{\infty} h(x(k), u(k)) &\le J_N(x(0)) + \left(\frac{\alpha_{N-1} - 1}{\alpha_{N-1}}\right) \sum_{k=1}^{\infty} J_N(x(k)) \\
&\le \left(1 + \left(\frac{\alpha_{N-1} - 1}{\alpha_{N-1}}\right) \sum_{k=1}^{\infty} \gamma_N^k\right) J_N(x(0))
\end{aligned}$$

which completes the proof. □

Corollary 3.1:

$$K_\infty = \{x_o : J(x_o) \le 1\}.$$

Proof: Clearly $J(x_o) \le 1$ implies $x_o \in K_\infty$.

Theorem 3.3 bounds the achievable infinite-horizon performance for any $x_o \in K_\infty$. Taking the limit of these bounds leads to

$$\lim_{N \to \infty} 1 + \left(\frac{\alpha_{N-1} - 1}{\alpha_{N-1}}\right)\frac{\gamma_N}{1 - \gamma_N} = 1$$

which implies that $J(x_o) \le 1$. □

Since the quantity $\frac{\alpha_{N-1}-1}{\alpha_{N-1}} \frac{\gamma_N}{1-\gamma_N}$ can be computed via finite-horizon computations, we can determine whether a receding horizon policy is sufficiently close to optimal *without* knowing the optimal cost.

IV. A Numerical Example

Consider the unstable system

$$x(k+1) = \begin{pmatrix} 1 & 1.1 \\ -1.1 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k)$$

with penalty function

$$h(x, u) = \left| \begin{pmatrix} x \\ u \end{pmatrix} \right|_\infty.$$

Table I presents the results of the finite-horizon computations. The "stability" column tabulates

$$\underline{\beta}_0 - (\alpha_{N-1} - 1)\bar{\beta}_N$$

which must be positive for stability (cf., Theorem 3.2). The "performance" column tabulates

$$\frac{\alpha_{N-1} - 1}{\alpha_{N-1}} \frac{\gamma_N}{1 - \gamma_N}$$

which bounds the infinite-horizon performance of the receding horizon policy (cf., Theorem 3.3).

We see that stability is achieved for $N \ge 4$, for which the receding horizon policy is at most 30% larger than the optimal infinite-horizon policy. For $N = 5$, we have $\alpha_4 = 1$, which implies that $K_4 \subset K_5$. Since $K_5 \subset K_4$ in general, it follows that $K_4 = K_5 = K_\infty$, and

TABLE I
FINITE-HORIZON COMPUTATIONS

N	α_{N-1}	β_N	Stability	Performance
1	3.1	3.10	-5.51	—
2	1.72	4.21	-2.01	—
3	1.28	5.39	-0.51	—
4	1.05	5.39	0.73	0.30
5	1	5.39	1	0

the receding horizon policy for $N = 5$ is in fact optimal. The set K_5 is described by ten inequalities

$$M_5 x = \begin{pmatrix} 3.0091 & 2.2100 \\ 2.7590 & 0.6300 \\ 1.9690 & 3.4200 \\ 2.2000 & 0.2100 \\ 1.4196 & 3.7715 \\ \vdots & \vdots \end{pmatrix} x \leq \mathbf{1}$$

where the bottom five inequalities are symmetric to the top five, i.e., the bottom five inequalities are the negatives of the top five.

The receding horizon control policy is

$$\phi(x;5) = \arg\min_v (\max_i (\Gamma_X x + \Gamma_U v)_i + \max_j (M_5 A x + M_5 B v)_j).$$

This optimization (which does not uniquely define v) does not need to be solved on-line. Let

$$M^{\text{temp}} = \text{Pair}[(\Gamma_X \quad \Gamma_U),\ (M_5 A \quad M_5 B)].$$

One can show via Proposition A.2 that there exist vectors $v_i \in \mathcal{R}^2$ and scalars w_i (from selected rows of M^{temp}) so that

$$\phi(x;5) = \max_{1 \leq i \leq 18} \frac{J_5(x) - v_i^T x}{w_i}.$$

Note that $\phi(\cdot;5)$ is Lipshitz continuous.

APPENDIX
CONSTRUCTION OF THE Rack[·] OPERATOR

In this Appendix, we review the construction of the Rack[·] operator.

Proposition A.1 [4]: Let $M = (M_1 \quad M_2)$, with $M_1 \in \mathcal{R}^{z \times n}$ and $M_2 \in \mathcal{R}^z$. Let

$$Z_+ = \{i : (M_2)_i > 0\}$$
$$Z_- = \{i : (M_2)_i < 0\}$$
$$Z_0 = \{i : (M_2)_i = 0\}.$$

Let $\tilde{M}$ be the matrix formed by the rows:

- $$\rho_{i_+ i_-}^T = \frac{1}{(M_2)_{i_+} - (M_2)_{i_-}} ((M_2)_{i_+}(M_1)_{(i_-,:)} - (M_2)_{i_-}(M_1)_{(i_+,:)}), \quad \forall i_+ \in Z_+, i_- \in Z_-;$$
- $\rho_{i_0}^T = (M_1)_{(i_0,:)}, \qquad \forall i_0 \in Z_0.$

Then $\tilde{M} = \text{Rack}[M]$.

Typically, $\tilde{M}$ contains several rows which represent redundant constraints and can be removed as needed.

Proposition A.2: In the framework of Proposition A.1, let Z_+ and Z_- be nonempty. Define $\psi_+^M : \mathcal{R}^n \to \mathcal{R}$ and $\psi_-^M : \mathcal{R}^n \to \mathcal{R}$ by

$$\psi_+^M(x) = \min_{i_+ \in Z_+} \frac{1 - (M_1)_{(i_+,:)} x}{(M_2)_{i_+}}$$
$$\psi_-^M(x) = \max_{i_- \in Z_-} \frac{1 - (M_1)_{(i_-,:)} x}{(M_2)_{i_-}}.$$

Then ψ_+^M and ψ_-^M are continuous and satisfy

$$(x, \psi_+^M(x)) \in \text{Set}(M), \qquad \forall x \in \text{Set}(\tilde{M})$$
$$(x, \psi_-^M(x)) \in \text{Set}(M), \qquad \forall x \in \text{Set}(\tilde{M}).$$

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