ELSEVIER

Systems & Control Letters 35 (1998) 1–11

SYSTEMS & CONTROL LETTERS

Output feedback control for systems with constraints and saturations: scalar control case[1]

Jeff S. Shamma *, Kuang-Yang Tu

Department of Aerospace Engineering and Engineering Mechanics, The University of Texas at Austin, Austin, TX 78712, USA

Received 24 March 1997; received in revised form 23 June 1997

Abstract

We consider the problem of regulating the state of a linear system within a prescribed region of the state-space in the presence of disturbances and control saturations. While prior work has focused on full state-feedback, we consider the case of noisy output feedback. For the scalar control case, we combine existing work on full state-feedback and set-valued observers to derive a computational procedure which determines a priori whether the desired constrained regulation is possible. Furthermore, we show that constrained regulation can always be achieved by a set-valued observer followed by static feedback. © 1998 Published by Elsevier Science B.V. All rights reserved.

Keywords: Saturation; Constrained regulation; Set-valued estimation

1. Introduction

All physical control systems are subject to constraints on the controlled variables and saturations on control authority. In the case of linear models which are derived from linearizations of nonlinear models, these constraints are typically handled implicitly in the design process and are verified afterwards by simulations.

There is a large body of work on explicitly addressing saturations and constraints. Various approaches include heuristic governing mechanisms augmented to linear control designs (cf. [12, 16]), nonlinear control designs for systems with special internal structures (cf. [24]), or explicit numerical optimization (cf. [19]).

So-called "set-valued methods" also have been applied to systems with saturations. These methods (cf. [2, 5–7, 9, 13–15, 21]) are based on the construction of a maximal controlled invariant set, i.e., a subset of the state-space will always contain the state under appropriate saturated state feedback. These methods essentially produce optimal (nonlinear) state feedback in the presence of constraints, saturations, and even disturbances.

This paper also uses set-valued methods, but focuses on the case of *noisy output feedback*. The approach is to merge set-valued state feedback methods with set-valued state observation methods to construct controllers which achieve a certain level of performance in the presence of process disturbances and measurement noises while conforming to prescribed state and control constraints.

A set-valued observer (SVO) produces a set-valued state estimate based on a priori disturbance/noise models and the system dynamics. The SVO is an outcome

---

[1] Supported by NSF grant #ECS-9258005, EPRI grant #8030-23, and AFOSR grant #F49620-97-1-0197.

* Corresponding author.

PII: S0167-6911(98)00013-9

of the "unknown-but-bounded" approach to estimation [3, 4, 20], which has received considerable attention in the controls literature. Refs. [8, 18] present an overview of work in this area, and Ref. [17] contains a collection of related conference papers.

The relationship between SVOs and output feedback control of constrained systems can be explained as follows. In the case of state feedback, one can determine whether it is possible to satisfy prescribed constraints in the presence of process disturbances. If so, each state vector has a corresponding *set* of control values which can be applied while guaranteeing that constraints will not be violated in the future. The relationship between the state and set of admissible controls is called the "regulation map" [1].

Now in the output feedback case, the state is not available. However, the SVO produces a set-valued state estimate, i.e., a set of possible states. The idea is then to select a control value which lies in the intersection of all regulation maps over the current set-valued state estimate. If this intersection is empty, then *no* control value assures that constraints will not be violated.

In this paper, it is shown (for the scalar control case) how to determine a priori whether the intersection of regulation maps is always non-empty. In this case, one controller which achieves the desired performance consists of a SVO combined with a static "selection strategy".

The remainder of this paper is organized as follows. Section 2 establishes notation and presents some preliminary results. Section 3 presents the problem formulation, reviews regulation maps and SVO's, and gives the main results. Section 4 contains numerical examples. Finally, Section 5 presents some concluding remarks.

2. Mathematical preliminaries

Some special notation will be used regarding representations of convex sets in $\mathbb{R}^n$. Boldface **1** denotes a column vector of 1's. For $M \in \mathbb{R}^{p\times n}$, define

$$Set(M) = \{x \in \mathbb{R}^n\colon Mx \leqslant \mathbf{1}\}.$$

The following property is particular to convex sets in $\mathbb{R}^2$.

Lemma 2.1. *Let S be a compact convex subset of* $\mathbb{R}^2$. *Define*

$$\underline{x}_1 = \min\left\{x_1\colon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S\right\},$$

$$\bar{x}_1 = \max\left\{x_1\colon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S\right\},$$

$$\underline{x}_2 = \min\left\{x_2\colon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S\right\},$$

$$\bar{x}_2 = \max\left\{x_2\colon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S\right\}.$$

Then

$$((\bar{x}_1 + \underline{x}_1)/2\ (\bar{x}_2 + \underline{x}_2)/2)^{\mathrm{T}} \in S.$$

Lemma 2.1 states that the Tchebychev center of a convex set, S, lies within S provided that $S \subset \mathbb{R}^2$. This is *not* the case for higher dimensions.

Corollary 2.1. *Let S be a compact subset of* $\mathbb{R}^n$. *For vectors* $\alpha_1, \alpha_2 \in \mathbb{R}^n$, *define*

$$\underline{a}_1 = \min\{\alpha_1^{\mathrm{T}}x\colon x \in S\}, \qquad \bar{a}_1 = \max\{\alpha_1^{\mathrm{T}}x\colon x \in S\},$$

$$\underline{a}_2 = \min\{\alpha_2^{\mathrm{T}}x\colon x \in S\}, \qquad \bar{a}_2 = \max\{\alpha_2^{\mathrm{T}}x\colon x \in S\}.$$

There exists an $x^* \in S$ *such that*

$$\alpha_1^{\mathrm{T}}x^* = (\bar{a}_1 + \underline{a}_1)/2, \qquad \alpha_2^{\mathrm{T}}x^* = (\bar{a}_2 + \underline{a}_2)/2.$$

3. Constrained regulation

3.1. Problem formulation

We will consider discrete-time systems of the form

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 d(k) + B_2 u(k),\\ z(k) &= C_1 x(k) + D_{12} u(k), \qquad (1)\\ y(k) &= C_2 x(k) + D_{21} n(k). \end{aligned}$$

with state $x(k) \in \mathbb{R}^n$ and control $u(k) \in \mathbb{R}^m$. The state equations are subject to process disturbances, $d(k)$, while available measurements, $y(k)$, are subject to measurement noises, $n(k)$. All signals are vector valued unless stated otherwise. (In later sections, we will consider only the scalar control case, $m = 1$.)

The following assumption is made on the disturbances and noises *throughout.*

Assumption 3.1. *For all $k \geqslant 0$,*

$$|d(k)| \leqslant 1, \qquad |n(k)| \leqslant 1.$$

Assumption 3.1 reflects that the disturbances and noises are uniformly bounded signals with a known normalized bound.

The vector, $z(k)$, reflects the constraints under which the system (1) must operate, namely

$$|z(k)| = |C_1 x(k) + D_{12} u(k)| \leqslant 1. \tag{2}$$

By appropriately defining C_1 and D_{12}, the constraint (2) can reflect polytopic bounds on the state variables as well as saturation bounds on the control variables.

We will admit fairly general controller structures. We will say a controller is any operator which maps a vector $x_0 \in \mathbb{R}^n$ and output sequence $\{y(0), y(1), y(2), \ldots\}$ into a control sequence $\{u(0), u(1), u(2), \ldots\}$ in a causal manner (cf. [25] for background on input/output operators, causality, and related issues). This relationship is denoted $u = \mathscr{K}[x_0]y$. The vector x_0 is used to initialize the controller, and may be viewed as an approximate initial condition for (1).

We now define the objective of constrained regulation.

Definition 3.1 (*Constrained regulation*)**.** Let S and E be compact sets in $\mathbb{R}^n$ with $0 \in E \subset S$. A controller, $\mathscr{K}$, achieves *constrained regulation* over (S, E) if for any $x_0 \in S$ and any initial condition $x(0) \in (x_0 + E) \cap S$, all solutions to Eq. (1) under the feedback $u = \mathscr{K}[x_0]y$ meet constraints (2) and satisfy $x(k) \in S$.

The set S represents a desired domain of operation for Eq. (1), while the set E represents uncertainty in the controller's knowledge of the initial condition of Eq. (1).

3.2. Background

Before proceeding with the discussion on constrained regulation, we will review some background material regarding (1) regulation maps and (2) set-valued observers.

We begin with the following definitions.

Definition 3.2 (*Constrained controlled invariance*)**.** A compact set $S \subset \mathbb{R}^n$ is *constrained controlled invariant* if constrained regulation over $(S, 0)$ is possible under full state feedback, i.e., $C_2 = I$, $D_{21} = 0$.

Definition 3.3 (*Regulation map*)**.** Let S be constrained controlled invariant. For any $x \in S$, let $R(x; S) \subset \mathbb{R}^m$ denote all control values, u, which meet constraints (2) while maintaining

$$Ax + B_1 d + B_2 u \in S, \quad \forall |d| \leqslant 1.$$

The *regulation map* is the set-valued map relating x and $R(x; S)$.

The regulation map [1] plays a central role in constrained regulation. In the full state feedback case, the regulation map characterizes *all* controllers which achieve constrained regulation over $(S, 0)$, namely, $u(k) \in R(x(k); S)$ for all k.

It is often possible to provide an explicit representation of the regulation map as follows.

Proposition 3.1. *Let S be a constrained controlled invariant set with matrix representation $S = Set(\binom{M}{-M})$, for some $M \in \mathbb{R}^{q \times n}$. In the scalar control case, $m = 1$, there exist vectors $\alpha_i \in \mathbb{R}^n$ and scalars $\beta_i \geqslant 0$ such that the regulation map $R(x; S)$ has the following representation*:

$$u \in R(x; S)$$

$$\Leftrightarrow$$

$$\max_i \alpha_i^{\mathrm{T}} x - \beta_i \leqslant u \leqslant \min_j \alpha_j^{\mathrm{T}} x + \beta_j.$$

As mentioned in the introduction, constrained controlled invariance has been considered by many authors, and algorithms exist which approximate maximal constrained controlled invariant sets via matrix representations.

We now turn our attention towards the set-valued observer (SVO). The SVO uses the a priori bounds on exogenous noises and disturbances to produce a set-valued estimate of state. This set-valued state estimate, denoted $X(k) \subset \mathbb{R}^n$, characterizes all states which are consistent with the current measurement trajectory.

To construct an SVO, we first define the set $\tilde{X}(k) \subset \mathbb{R}^n$ by

$$\tilde{X}(k) = \{x \in \mathbb{R}^n\colon\ y(k) = C_2 x(k) + D_{21} n \text{ for some } |n| \leqslant 1\}.$$

The set $\tilde{X}(k)$ represents all possible states at time k based on the single measurement $y(k)$ only. Similarly, define

$$X_{pre}(k+1) = \{x\colon\ x = A\tilde{x} + B_1 d + B_2 u(k),\quad \text{for some } \tilde{x}\in X(k),\ |d|\leqslant 1\}.$$

The set $X_{pre}(k+1)$ denotes the *anticipated* set of possible states at time $k+1$ based on measurements up to time k. Note that X, $\tilde{X}$, and X_{pre} all depend on the current measurement trajectory and current control input. However, this dependence is not explicitly expressed for the sake of notational simplicity,

Algorithm 3.1 (SVO). *Let* $y = \{y(0), y(1), y(2), \ldots\}$ *and* $u = \{u(0), u(1), u(2), \ldots\}$ *be a measurement trajectory and control trajectory, respectively, of the system* (1). *Suppose* $x(0)\in X_0$.

Initialization

$$X_{pre}(0) = X_0,$$

$$X(0) = X_{pre}(0)\cap\tilde{X}(0).$$

Propagation

$$\begin{aligned} X(k) &= X_{pre}(k)\cap\tilde{X}(k)\\ &= \{x\colon\ x = A\tilde{x} + B_1 d + B_2 u(k),\\ &\qquad \text{for some } \tilde{x}\in X(k-1), |d|\leqslant 1\}\cap\tilde{X}(k). \end{aligned}$$

Note that the compactness assumption on E assures that the sets $X(k)$ are also compact.

Refs. [22, 23] discuss the computation of matrix representations

$$X(k) = \{x\colon\ M(k)x\leqslant m(k)\}$$

to explicitly construct the $X(k)$.

Associated with SVOs are so-called "central estimates". Let

$$v(k) = Hx(k)$$

be a scalar variable which is to be estimated. Suppose that at time k, the state is known to lie within the set $X(k)$. Let

$$\underline{v}(k) = \min_{x\in X(k)} Hx,$$

$$\overline{v}(k) = \max_{x\in X(k)} Hx.$$

Then $v(k)$ is known to lie in the interval $[\underline{v}(k), \overline{v}(k)]$. Now define the central estimate

$$\hat{v}_c(k) = \frac{1}{2}(\overline{v}(k) + \underline{v}(k)) \tag{3}$$

as the center of this interval.

Clearly the central estimate, $\hat{v}(k)$, is the optimal estimate of $v(k)$ in a minimax sense [18]. Refs. [23, 23] further discuss optimality properties of central estimates and shows that central estimates are optimal in an induced norm sense as well.

Define $\mathcal{O}(k)$ as the set of reachable states at time k (with admissible disturbances and noises) subject to $x(0) = 0$, $u = 0$, and

$$y(0) = 0, \ldots, y(k) = 0.$$

The set $\mathcal{O}(k)$ represents all reachable states from zero initial conditions such that the output is identically zero. Now define

$$\mathcal{O} = \overline{\bigcup_{k=0}^{\infty}\mathcal{O}(k)}, \tag{4}$$

i.e., the closure of the infinite union of the individual $\mathcal{O}(k)$.

From the above definition of $\mathcal{O}$, it is clear that

$$\max_{x\in\mathcal{O}} |Hx|$$

is a *lower bound* on the achievable performance for any estimate of v. This can be seen by setting $x(0) = 0$. The disturbances and noises can combine to produce $y = 0$ while driving the state anywhere within $\mathcal{O}$.

Proposition 3.2 (Shamma and Tu [22, 23]). *Let* $v(k) = Hx(k)$ *and define the central estimate* $\hat{v}(k)$ *as in* (3). *Consider the SVO initialized with*

$$X(0) = x_0 + \mathcal{O},$$

for some $x_0\in\mathbb{R}^n$. *For any initial condition* $x(0)\in X(0)$ *and any admissible disturbance/noise trajectories, the central estimate satisfies*

$$|v(k) - \hat{v}_c(k)|\leqslant \max_{x\in\mathcal{O}} |Hx|.$$

Proposition 3.2 states that the central estimate, $\hat{v}_c$, achieves the optimal level of performance. In fact, Refs. [22, 23] shows that the estimation the SVO is optimal for the *current* measurement trajectory, and hence its estimation error in fact may be less than that stated in Proposition 3.2, which is a worst-case bound.

3.3. Main results

We now consider constrained regulation under output feedback.

Our starting point is to *assume* the existence of a compact set, $S \subset \mathbb{R}^n$, such that

(S1) $S = Set\left(\begin{pmatrix} M \\ -M \end{pmatrix}\right)$ for some $M \in \mathbb{R}^{q\times n}$.
(S2) $\mathcal{O} \subset S$.
(S3) S is constrained controlled invariant.

Our goal is to obtain conditions under which there exists an *output* feedback controller which achieves constrained regulation over $(S, \mathcal{O})$.

The matrix representation of S causes minimal loss of generality. This representation is consistent with existing algorithms which can computationally approximate maximal constrained controlled invariant sets to within any desired degree of accuracy (cf. discussion and references in [21]).

The condition that $\mathcal{O} \subset S$ is a natural consequence of the definition of $\mathcal{O}$. Namely, starting from zero initial conditions, the state can lie anywhere in $\mathcal{O}$ without providing information to the controller ($y = 0$). Therefore, it is impossible to enforce constrained regulation over any set which does not contain $\mathcal{O}$.

The following theorem presents conditions for constrained regulation over $(S, \mathcal{O})$ in terms of the regulation map.

Theorem 3.1. *A controller*, $\mathcal{K}$, *does* not *achieve constrained regulation over* $(S, \mathcal{O})$ *if there exists a trajectory of* (1) *under the feedback* $u = \mathcal{K}[x_0]y$ *such that the set-valued estimate*, $X(k)$, *satisfies*

$$\bigcap_{x\in X(k)} R(x; S) = \emptyset \tag{5}$$

for some time k.

Theorem 3.1 states that if disturbances and noises can combine to cause (5), then an admissible disturbance trajectory can drive the state out of S. Conversely, if the state vector is to remain in S for *all* admissible disturbances/noises, then the control value at any time, k, *must* satisfy $u(k) \in R(x(k); S)$. In the output feedback case, the state $x(k)$ is known only to lie anywhere within the set-valued estimate $X(k)$. Therefore, a *single* control value must suffice for all $x(k) \in X(k)$. Note that these conditions hold for all controllers.

Theorem 3.1 does not provide an *a priori* test on whether constrained regulation is possible. However, it does characterize the relationship between the state feedback problem and the estimation problem.

We are now in a position to state the main result.

Theorem 3.2. *Let* S *be as in conditions* (S1)–(S3). *Let* $R(x; S)$ *be representation by parameters* (α_i, β_i) *as in Proposition* 3.1. *Define the scalar parameters*

$$e_i = \max\{\alpha_i^{\mathrm{T}} x\colon x \in \mathcal{O}\}, \tag{6}$$

$$a_i = \max\{\alpha_i^{\mathrm{T}} x\colon x \in S\}. \tag{7}$$

There exists an output feedback controller which achieves constrained regulation over $(S, \mathcal{O})$ *if and only if for all* i, j *and all* $x \in S$,

$$\beta_i - e_i \geqslant 0, \tag{8}$$

$$(\alpha_i^{\mathrm{T}} - \alpha_j^{\mathrm{T}})x \leqslant \max\{\beta_i - e_i, \beta_i - (a_i - \alpha_i^{\mathrm{T}} x)\} + \max\{\beta_j - e_j, \beta_j - (a_j + \alpha_j^{\mathrm{T}} x)\}. \tag{9}$$

The scalars e_i and a_i represent worst-case estimation errors for $\alpha_i^{\mathrm{T}} x$, and maximal values of $\alpha_i^{\mathrm{T}} x$, respectively.

It is straightforward to check computationally the conditions of Theorem 3.2 by solving appropriate linear programs.

Theorem 3.2 states that one can determine whether constrained regulation under output feedback is possible by examining the worst-case estimation errors associated with (1) the vectors α_i and (2) the vector *differences* $\alpha_i - \alpha_j$. For example, suppose that the regulation map is characterized by exactly one inequality.

$$\alpha_1^{\mathrm{T}} x - \beta_1 \leqslant u \leqslant \alpha_1^{\mathrm{T}} x + \beta_1.$$

In this case, constrained regulation is achieved by the state feedback $u(k) = \alpha_1^{\mathrm{T}} x(k)$. Theorem 3.2 states that constrained regulation under output feedback depends on the estimation error associated with $\alpha_1^{\mathrm{T}} x$. Loosely speaking, the optimal output feedback consists of an optimal estimate of the optimal full state feedback, which is reminiscent of separation structures in other types of optimal disturbance rejection [11].

The set $\mathcal{O}$ is a natural uncertainty set for initial conditions. In the case of smaller uncertainty sets, $E \subset \mathcal{O}$, conditions (8) and (9) remain sufficient for constrained regulation over (S, E), and condition (8) remains necessary. It is unclear whether condition (9) remains necessary as well.

3.4. Proof of main result

This section is devoted to the proof of Theorem 3.2.

Proof. (If) we will show that constrained regulation over $(S, \mathcal{O})$ can be achieved by an SVO combined with a static selection strategy. The SVO, initialized with $X_{pre}(0) = (x_0 + \mathcal{O}) \cap S$, will construct set-valued estimates, $X(k)$. Given $X(k)$, the control value $u(k)$ can be any selection in the intersection of regulation maps over $X(k)$, i.e.,

$$u(k) \in \bigcap_{x \in X(k)} R(x). \tag{10}$$

We will show that conditions (8) and (9) in Theorem 3.2 assure that the above intersection is non-empty whenever $X(k) \subset S$.

Let $X(k)$ be a particular set-valued state estimate with $X(k) \subset S$. We require that for all i, j,

$$\max_{x \in X(k)} \alpha_i^{\mathrm{T}} x - \beta_i \leqslant \min_{x \in X(k)} \alpha_j^{\mathrm{T}} x + \beta_j. \tag{11}$$

According to Proposition 3.2, the scalar e_i represents the worst-case estimation error for $\alpha_i^{\mathrm{T}} x$. Thus for any $x \in X(k)$, $\alpha_i^{\mathrm{T}} x$ satisfies

$$\alpha_i^{\mathrm{T}} x \in [\widehat{(\alpha_i^{\mathrm{T}} x)}_c - e_i, \widehat{(\alpha_i^{\mathrm{T}} x)}_c + e_i] \cap [-a_i, a_i],$$

where $\widehat{(\alpha_i^{\mathrm{T}} x)}_c$ represents the central estimate of $\alpha_i^{\mathrm{T}} x$ based on $X(k)$. The intersection with the interval $[-a_i, a_i]$ reflects the uniform bound on $\alpha_i^{\mathrm{T}} x$ over S. Using these bounds on $\alpha_i^{\mathrm{T}} x$, leads to

$$\max_{x \in X(k)} \alpha_i^{\mathrm{T}} x \leqslant \widehat{(\alpha_i^{\mathrm{T}} x)}_c + \min\{e_i, a_i - \widehat{(\alpha_i^{\mathrm{T}} x)}_c\}.$$

Similarly,

$$\min_{x \in X(k)} \alpha_j^{\mathrm{T}} x \geqslant \widehat{(\alpha_j^{\mathrm{T}} x)}_c - \min\{e_j, a_j + \widehat{(\alpha_j^{\mathrm{T}} x)}_c\}.$$

Therefore, the regulation map intersection is non-empty provided that for all i, j,

$$\begin{aligned}\widehat{(\alpha_i^{\mathrm{T}} x)}_c + \min\{e_i, a_i - \widehat{(\alpha_i^{\mathrm{T}} x)}_c\} - \beta_i &\leqslant \widehat{(\alpha_j^{\mathrm{T}} x)}_c \\ &\quad - \min\{e_j, a_j + \widehat{(\alpha_j^{\mathrm{T}} x)}_c\} + \beta_j.\end{aligned} \tag{12}$$

Corollary 2.1 assures that there exists an $x^* \in X(k)$ (which depends on i, j, and $X(k)$) such that

$$\widehat{(\alpha_i^{\mathrm{T}} x)}_c = \alpha_i^{\mathrm{T}} x^*,$$

$$\widehat{(\alpha_j^{\mathrm{T}} x)}_c = \alpha_j^{\mathrm{T}} x^*.$$

Substituting x^* into Eq. (12) leads to

$$\begin{aligned}(\alpha_i^{\mathrm{T}} - \alpha_j^{\mathrm{T}}) x^* \leqslant \beta_i - \min\{e_i, a_i - \alpha_i^{\mathrm{T}} x^*\} + \beta_j \\ - \min\{e_j, a_j + \alpha_j^{\mathrm{T}} x^*\},\end{aligned}$$

which is equivalent to the hypothesis (9) in Theorem 3.2. This implies that the regulation map intersection over $X(k)$ is non-empty, as desired. Selecting $u(k)$ as in (10) assures that $X(k+1) \subset S$. Proceeding by induction (and noting that $X(0) \subset S$) leads to the desired result.

(Only if). Theorem 3.1 states that the regulation map must have non-empty intersection over *any* possible $X(k)$. This includes all initial set-valued estimates $X(0)$. We will show that conditions (8) and (9) follow from imposing various initial conditions on Eq. (1).

We first note the following consequence of the definition of $\mathcal{O}$. Suppose that in the SVO,

$$X_{pre}(0) = (x_0 + \mathcal{O}) \cap S,$$

and $x(0) = x_0$. Then

$$X(0) = (x_0 + \mathcal{O}) \cap S,$$

i.e., the measurement at $k = 0$ does not provide additional information.

Condition (8) of Theorem 3.2 follows from the initial condition uncertainty set, $X_{pre}(0) = \mathcal{O}$, and initial condition $x(0) = 0$. In this case, the information available to *any* controller at time $k = 0$ is that $x(0) \in \mathcal{O}$. Substituting $k = 0$, $X(k) = \mathcal{O}$, and $i = j$ into (11) leads to condition (8). Note that since $\mathcal{O}$ is symmetric with respect to the origin,

$$\max_{x \in \mathcal{O}} \alpha_i^{\mathrm{T}} x = -\min_{x \in \mathcal{O}} \alpha_i^{\mathrm{T}} x.$$

Condition (9) of Theorem 3.2 similarly follows from the initial condition uncertainty set, $X_{pre}(0) = (x_0 + \mathcal{O}) \cap S$, and initial condition, $x(0) = x_0$, for various $x_0 \in S$. Once again, the information available to any controller at time $k = 0$ is that $x(0) \in (x_0 + \mathcal{O}) \cap S$. In this case, arguments similar to the "if" proof show that the non-empty intersection requirement on the regulation map implies condition (9). □

3.5. Discussion

Theorem 3.2 states conditions under which constrained regulation under output feedback is possible. This implicitly assumes that constrained regulation under full state feedback is possible. Therefore, a first step towards applying Theorem 3.2 is the construction of a set over which constrained regulation is possible via full state feedback.

Clearly any set which is constrained controlled invariant must lie within the set

$$S_0 \overset{\text{def}}{=} \{x\colon\ |C_1x + D_{12}u| \leqslant 1,\ \text{for some } u\}.$$

As mentioned earlier, existing algorithms can be used to (approximately) construct the maximal constrained controlled invariant set within S_0.

Suppose S^* is a constrained controlled invariant set constructed with such methods, and suppose that one of the conditions of Theorem 3.2 is violated.

If condition (9) is violated, it may be that S^* is simply "too large", i.e., there exists a smaller subset $S_{\text{inner}} \subset S^*$ such that constrained regulation with output feedback is possible. This often tends to be the case, since the aforementioned state feedback algorithms construct "maximal" sets.

If condition (8) is violated, then *no* subset of S^* can lead to constrained controlled invariance. This is because attempting controlled invariance over some $S_{\text{inner}} \subset S^*$ will effectively lead to *smaller* β_i's, and hence condition (8) will always be violated.

Computational experience has shown that the following algorithm can be used to successfully apply Theorem 3.2:

- Initialize $S_{\text{test}} = S_0$.
- Construct (approximately) maximal controlled invariant subset $S^* \subset S_{\text{test}}$ and associated regulation map $R(x; S^*)$.
- Test conditions (8) and (9) of Theorem 3.2:
  - If conditions (8) and (9) pass, then constrained regulation over $(S^*, \mathcal{O})$ is possible.
  - If condition (8) fails, constrained regulation over $(S^*, \mathcal{O})$ is impossible.
  - If condition (9) fails, *redefine* S_{test} as S^* *plus* additional constraints

    $$(\alpha_i^{\mathrm{T}} - \alpha_i^{\mathrm{T}})x \leqslant \beta_i - e_i + \beta_j - e_j$$

    and repeat.

It is conjectured that the above iterations can determine definitely whether or not constrained regulation over a subset of S_0 is possible. The iterations have successfully predicted ℓ^1 optimal performance levels under output feedback for examples presented in [10].

A final comment regarding the application of Theorem 3.2 is that the estimation errors, e_i, need not be evaluated by explicitly computing $\mathcal{O}$. Rather, a linear program can be used to maximize directly (over bounded disturbances/noises) the cost $\alpha_i^{\mathrm{T}}x$ subject to the measured output being identically zero.

4. Numerical examples

4.1. Constrained stabilization

Consider the second order system,

$$x(k+1) = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix} x(k) + \frac{1}{\gamma}\begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix} d(k) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u(k),$$

$$z(k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} u(k),$$

$$y(k) = (1 \quad -1)x(k) + \frac{1}{\gamma}0.1n(k).$$

Note that the dynamics are unstable and non-minimum phase (from u to y).

The control objective $|z(k)| \leqslant 1$ reflects (1) desired state bounds, (2) desired control bounds, and (3) a "mixed" state/control bound.

The scalar γ is a normalization parameter. A small γ implies increased tolerance to disturbances and noises.

State-feedback constrained controlled invariant sets were constructed as suggested in Section 3.5 and then tested for output feedback constrained regulation. The maximal disturbance/noise tolerance level (i.e., minimal γ) was found to be $\gamma \simeq 5.2$. For this value of γ, it is possible to achieve constrained regulation over sets $(S^*, \mathcal{O})$, where S is shown in Fig. 1, and $\mathcal{O}$ in Fig. 2.

The regulation map representation of Proposition 3.1 is given by

$$\begin{pmatrix} \alpha_1^{\mathrm{T}} \\ \vdots \\ \alpha_5^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} -1.0000 & -1.0000 \\ 0 & 0 \\ 0 & -0.2500 \\ 6.6667 & -1.0833 \\ -2.6667 & 0.0833 \end{pmatrix},$$

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_5 \end{pmatrix} = \begin{pmatrix} 1.0000 \\ 1.0000 \\ 0.4903 \\ 1.7128 \\ 0.1342 \end{pmatrix}.$$

Theorem 3.2 assures that the regulation map will be non-empty for any admissible disturbance/noise

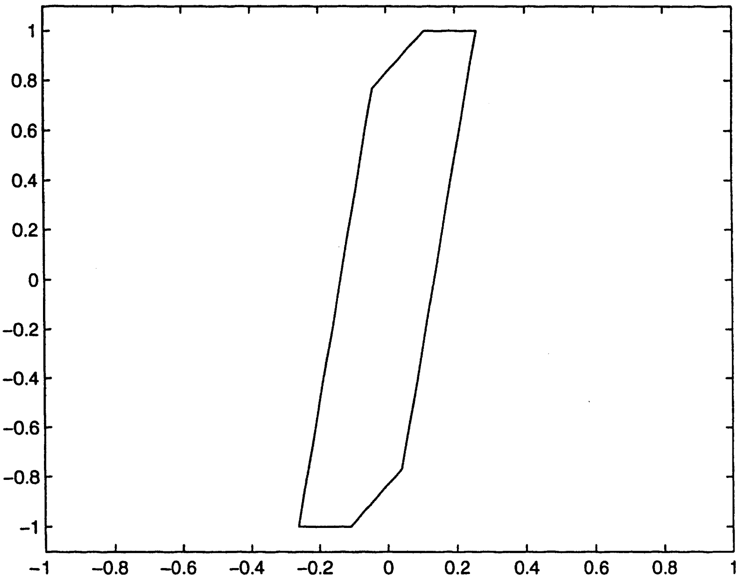

Fig. 1. Controlled invariant set for constrained stabilization.

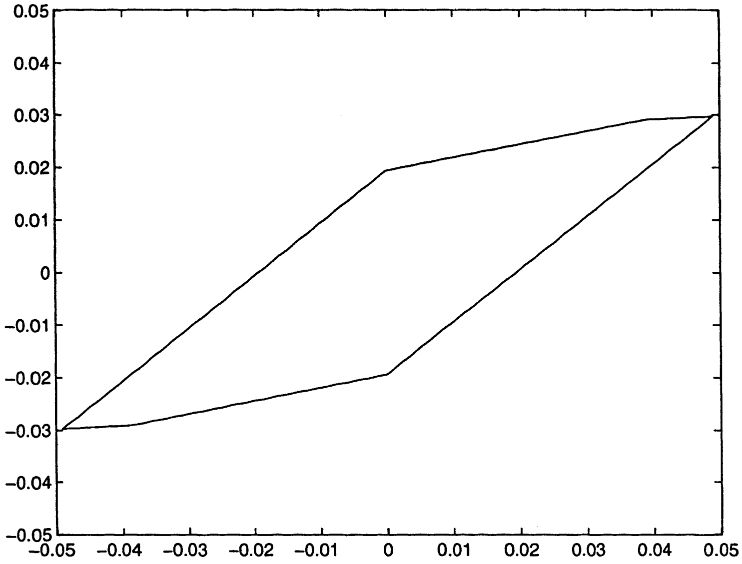

Fig. 2. Worst-case observation set $\mathcal{O}$ for constrained stabilization.

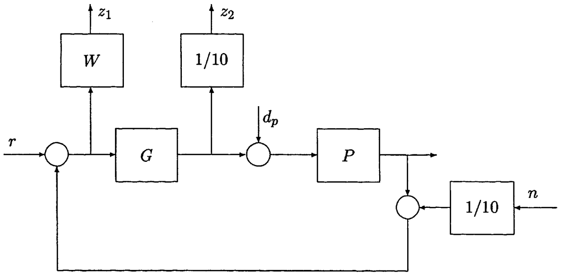

Fig. 3. Block digram for constrained command following.

trajectory and any initial condition uncertainty set of the form $x(0) \in (x_0 + \mathcal{O}) \cap S^*$ with $x_0 \in S^*$.

4.2. Constrained command following

The following example is adapted from [10] for ℓ_1-optimal control.

Consider the block diagram of Fig. 3 , where r represents a reference command; n, measurement noise; and d_p an process disturbance. The constraint signals are z_1, weighted tracking error, and $z_2 = u$, the scaled control.

The plant dynamics are given by

$$x_p(k+1) = \begin{pmatrix} 10.5 & -5 \\ 1 & 0 \end{pmatrix} x_p(k) + \frac{1}{\gamma}\begin{pmatrix} 1 \\ 0 \end{pmatrix} d_p(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k).$$

The available measurement is

$$y(k) = (5 \quad -10) x_p(k) + \frac{1}{\gamma} r(k) + \frac{1}{\gamma} 0.1 n(k).$$

The performance weighting, W, is the first order system

$$x_w(k+1) = 0.2 x_w(k) + 0.02 y(k),$$

$$z_1(k) = x_w(k).$$

Note that the plant is unstable and non-minimum phase (from u to y).

The control objective is to maintain $|z_{1,2}(k)| \leqslant 1$ for commands, disturbances, and noises satisfying $|r(k)|, |d(k)|, |n(k)| \leqslant 1/\gamma$, with γ to be made as small as possible. This is equivalent to minimizing weighted tracking error subject to control saturations, input disturbances, and measurement noise.

The plant, P, and performance weighting, W, together comprise a third order system. Two auxiliary states were introduced in order to convert the tracking problem to the constrained regulation form considered here, namely (1) process disturbances do not directly affect y and (2) measurement noises do not directly affect z. The resulting "extended" plant is fifth order with states, $x(k)$, representing

$$x_1(k) = x_w(k), \quad x_2(k) = x_{p,1}(k),$$
$$x_3(k) = x_{p,2}(k), \quad x_4(k) = n(k), \quad x_5(k) = r(k).$$

The minimal achieved γ under output feedback is $\gamma \simeq 8.35$. The final regulation map representation of Proposition 3.1 is given by

$$\begin{pmatrix} \alpha_1^{\mathrm{T}} \\ \vdots \\ \alpha_5^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -0.4000 & -8.7000 & 5.4000 & -0.0040 & -0.0400 \\ -0.0092 & -9.8839 & 5.0092 & -0.0001 & -0.0009 \\ 0 & -10.0000 & 5.0000 & 0 & 0 \\ 0.0615 & -10.7769 & 4.9385 & 0.0006 & 0.0062 \end{pmatrix},$$

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_5 \end{pmatrix} = \begin{pmatrix} 10.00 \\ 9.85 \\ 1.81 \\ 0.77 \\ 6.21 \end{pmatrix}.$$

Fig. 4 shows simulation results of a combined SVO and regulation map selection under a step command with random disturbances and noises. The step command and disturbances/noises continue until time $k = 50$, after which all exogenous inputs equal zero. The set-valued state estimates typically were represented by 10–12 constraints. Theoretically, the number of required constraints can increase without bound. However, "tolerance" levels may be set to keep the complexity in check.

5. Concluding remarks

We have seen that constrained regulation, whenever possible, can be achieved by a SVO in conjunction with a static selection strategy. This decomposition resembles a "separation structure" of a state observation combined with state feedback. Ref. [23] has recognized this decomposition as a general structure for constrained regulation problems, even in the multivariable control case. The problem

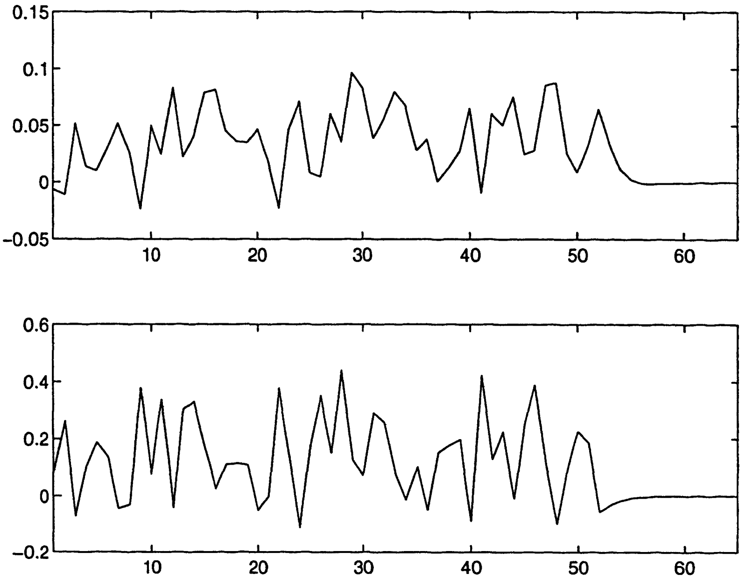

Fig. 4. Time responses for Constrained Command Following.

is then to determine a priori whether a certain level of performance is achievable.

The real-time computational burden of these methods (of producing set-valued estimates) currently limits their applicability to systems with slow dynamics, such as process control.

References

[1] J.P. Aubin, Viability Theory, Birkhäuser, Boston, 1991.

[2] D.P. Bertsekas, I.B. Rhodes, On the minimax reachability of target sets and target tubes, Automatica 7 (1971) 233–247.

[3] D.P. Bertsekas, I.B. Rhodes, Recursive state estimation for a set-membership description of uncertainty, IEEE Trans. Automat. Control AC-16 (1971).

[4] D.P. Bertsekas, I.B. Rhodes, Sufficiently informative functions and the minimax feedback control of uncertain dynamic systems, IEEE Trans. Automat. Control AC-18 (1973) 117–124.

[5] G. Bitsoris, E. Gravalou, Comparison principle, positive invariance and constrained regulation of nonlinear systems, Automatica 31 (1995) 217–222.

[6] G. Bitsoris, M. Vassilaki, Constrained regulation of linear systems, Automatica 31 (1995) 223–229.

[7] F. Blanchini, Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions, IEEE Trans. Automat. Control AC-39 (2) (1994) 428–433.

[8] F.L. Chernousko, State Estimation for Dynamic Systems, CRC Press, Boca Raton, FL, 1994.

[9] M. Cwikel, P.-O. Gutman, Convergence of an algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states, IEEE Trans. Automat. Control AC-31 (5) (1986) 457–459.

[10] M.A. Dahleh, I.J. Diaz-Bobillo, Control of Uncertain Systems: A Linear Programming Approach, Prentice-Hall, Englewood Cliffs, NJ, 1995.

[11] J.C. Doyle, K. Glover, P.P. Khargonekar, B. Francis, State-space solutions to standard $\mathscr{H}^2$ and $\mathscr{H}^\infty$ control problems, IEEE Trans. Automat. Control AC-34 (8) (1989) 821–830.

[12] E.G. Gilbert, K.T. Tan, Linear systems with state and control constraints: the theory and application of maximal output admissible sets, IEEE Trans. Automat. Control AC-36 (9) (1991) 1008–1020.

[13] P.-O. Gutman, M. Cwikel, Admissible sets and feedback control for discrete-time linear dynamical systems with bounded controls and states, IEEE Trans. Automat. Control AC-31 (4) (1986) 373–376.

[14] P.-O. Gutman, M. Cwikel, An algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states, IEEE Trans. Automat. Control AC-32 (3) (1987) 251–254.

[15] S.S. Keerthi, E.G. Gilbert, Computation of minimum-time feeedback control laws for discrete-time systems with state-control constraints, IEEE Trans. Automat. Control AC-32 (5) (1987) 432–435.

[16] M.V. Kothare, P.J. Campo, M. Morari, C.N. Nett, A unified framework for the study of anti-windup designs, Automatica 30 (12) (1994) 1869–1884.

[17] A.B. Kurzhanski, V.M. Veliov (Eds.) Modeling Techniques for Uncertain Systems, Birkhäuser, Boston, 1993.

[18] M. Milanese, V. Vicino, Optimal estimation theory for dynamic systems with set membership uncertainty: An overview, Automatica 27 (1991) 997–1009.

[19] J.B. Rawlings, K.R. Muske, The stability of constrained receding horizon control, IEEE Trans. Automat. Control AC-38 (10) (1993) 1512–1516.

[20] F.C. Schweppe, Uncertain Dynamic Systems, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[21] J.S. Shamma, Optimization of the ℓ^∞-induced norm under full state feedback, IEEE Trans. Automat. Control AC-41 (4) (1996) 533–544.

[22] J.S. Shamma, K.-Y. Tu, Optimality of set-valued observers for linear systems, in: Proc. 34th IEEE Conf. on Decision and Control, New Orleans, LA, December 1995.

[23] J.S. Shamma, K.-Y. Tu, Set-valued observers and optimal disturbance rejection, IEEE Trans. Automat. Control, to appear.

[24] A.R. Teel, A nonlinear small gain theorem for the analysis of control systems with saturation, IEEE Trans. Automat. Control AC-41 (9) (1996) 1295–1312.

[25] J.C. Willems, The Analysis of Feedback Systems, MIT Press, Cambridge, MA, 1971.