Existence of SDRE Stabilizing Feedback

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Abstract

The state-dependent Riccati equation (SDRE) approach to nonlinear system stabilization relies on representing a nonlinear system’s dynamics in a manner to resemble linear dynamics, but with state-dependent coefficient matrices that can then be inserted into state-dependent Riccati equations to generate a feedback law. Although stability of the resulting closed loop system need not be guaranteed a priori, simulation studies have shown that the method can often lead to suitable control laws. In this paper, we consider the non-uniqueness of such a representation. In particular, we show that if there exists any stabilizing feedback leading to a Lyapunov function with star-shaped level sets, then there always exists a representation of the dynamics such that the SDRE approach is stabilizing. The main tool in the proof is a novel application of the S-procedure for quadratic forms.

1 Overview

SDRE stabilization refers to the use of State Dependent Riccati Equations to construct nonlinear feedback control laws for nonlinear systems. This approach has received considerable attention in recent years [2, 3, 4]. The main idea is to represent the nonlinear system

\[ \dot{x} = f(x) + B(x)u \]  (1)

in the form

\[ \dot{x} = A(x)x + B(x)u \]  (2)

and to use the feedback

\[ u = -R^{-1}(x)B^T(x)P(x)x \]

where \( P(x) \) is obtained from the SDRE

\[ P(x)A(x) + A^T(x)P(x) + Q(x) - P(x)B(x)R^{-1}(x)B^T(x)P(x) = 0 \]

and \( Q(\cdot) \) and \( R(\cdot) \) are design parameters that satisfy the pointwise positive definiteness condition

\[ Q(x) > 0 \quad R(x) > 0 \]

The resulting closed loop dynamics have a quasi-linear structure

\[ \dot{x} = (A(x) - R^{-1}(x)B(x)B^T(x)P(x))x \]

where the “dynamics” matrix satisfies the pointwise Hurwitz condition

\[ \text{Re}\lambda_i (A(x) - R^{-1}(x)B(x)B^T(x)P(x)) < 0 \]

Although this condition is not sufficient to assure stability of the closed-loop dynamics, simulation studies have shown that this procedure is capable of deriving effective control laws.

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As mentioned earlier, the matrices $Q(\cdot)$ and $R(\cdot)$ are design parameters in the SDRE approach. Another non-obvious design parameter is the choice of the representation of the original dynamics (1) in the quasi-linear form (2). It is easy to see that

$$\dot{x} = (A(x) + E(x))x + B(x)u$$

is also a representation for any matrix that satisfies

$$E(x)x = 0$$

The following example is a simple illustrative example

**Example 1.1** Consider the nonlinear plant

$$\begin{align*}
\dot{x}_1 &= \sin(x_1) + x_2 \\
\dot{x}_2 &= x_1 x_2 + u
\end{align*}$$

(3)

Two equivalent representations are

$$\dot{x} = A(x)z + Bu = \begin{bmatrix} \sin(x_1)/x_1 & 1 \\ x_2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

and

$$\dot{x} = \begin{bmatrix} \sin(x_1)/x_1 & 1 \\ 0 & x_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The issue of non-uniqueness can play a major role, even affecting the controllability of the resulting parameterized pair $(A(x), B(x))$. Note that the presence or lack of controllability of this pair need not have any implication on the controllability of the original dynamics. These issues have been considered in [5].

References [4, 6] also considered the implications of non-uniqueness of the representation (2). Reference [4] derives a necessary condition on $f(x)$ and $B(x)$ for the existence of any feedback gain matrix, $G(x)$, that results in

$$A(x) + B(x)G(x)$$

being pointwise Hurwitz. Reference [6] shows that a class of nonlinear optimal control problems have the following property. The optimal feedback law

$$u = g(x)$$

can be written as

$$u = -R^{-1}(x)B^T(x)P(x)x$$

where $P(x)$ is the positive definite solution to the Riccati equation

$$P(x)(A(x) + E(x)) + (A(x) + E(x))^T P(x) + Q(x) - P(x)B(x)R^{-1}(x)B^T(x)P(x) = 0$$

for some $E(x)$ that satisfies

$$E(x)x = 0$$

In other words, there will exist a representation such that the SDRE feedback produces the optimal feedback law.

The following presentation follows the work of [6]. The main results are:

- If there exists any stabilizing feedback law, $u = g(x) = K(x)x$, that admits a Lyapunov function with “star-convex” level sets, then there exists a representation such that

$$A(x) + E(x) + B(x)K(x)$$

is pointwise Hurwitz, where $E(x)x = 0$.

- Under the same assumptions, there exists a representation, $A(x) + E(x)$, and pointwise positive definite matrices, $Q(x)$ and $R(x)$, such that the SDRE approach is stabilizing.

Loosely speaking, the implications are that if a system is stabilizable (under an appropriate Lyapunov function), then it is stabilizable via the SDRE method. The main tool in the proof is a novel application of the $S$-procedure for quadratic forms.

Unfortunately, these results do not suggest how to derive an appropriate representation.

**Notation**

For $x \in \mathbb{R}^n$, define

$$|x| = (x^T x)^{1/2}$$

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For a differentiable function $V(x)$ on $\mathbb{R}^n$, $DV(x)$ denotes the matrix of partial derivatives, and $D_iV(x)$ denotes the partial derivative with respect to the $i^{th}$ independent variable, $x_i$.

2 Preliminaries

We will need the S-procedure for quadratic forms [1]. Suppose that two symmetric $n \times n$ matrices, $M_0$ and $M_1$, satisfy the following:

- For $x \in \mathbb{R}^n$, $x^T M_1 x \geq 0 \Rightarrow x^T M_0 x > 0$
- There exists a $y \in \mathbb{R}^n$ such that $y^T M_1 y > 0$

Then there exists a $\tau \geq 0$ such that $M_0 - \tau M_1 > 0$

The following is a consequence of the S-procedure.

Proposition 2.1 Suppose that the $n \times n$ matrix, $P(x)$, satisfies

$$x^T P(x) x > 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}$$

Then there exists an $n \times n$ matrix $\tilde{P}(x)$ such that

$$P(x)x = \tilde{P}(x)x, \quad \forall x \in \mathbb{R}^n$$

and

$$v^T P(x)v > 0, \quad \forall v, x \in \mathbb{R}^n \setminus \{0\}$$

Proof Define

$$M_1(x) = x^T x I - xx^T$$

Note that

$$M_1(x)x = 0$$

Furthermore, for all $v$

$$v^T M_1(x)v = x^T x v^T v - v^T x x^T v$$

$$= |x|^2 |v|^2 - (v^T x)^2$$

$$\geq 0$$

Therefore, if for some $v$

$$v^T (-M_1(x))v \geq 0$$

then necessarily $v = \alpha x$ for some $\alpha \in \mathbb{R}$. This in turn implies that

$$v^T (-M_1(x))v \geq 0 \Rightarrow v^T P(x)v > 0$$

Via the S-procedure, for any $x \in \mathbb{R}^n \setminus \{0\}$,

$$P(x) + \tau(x) M_1(x) > 0$$

for some $\tau(x) \geq 0$.

3 Stabilization via Representation

We now consider the stabilization of the nonlinear system

$$\dot{x} = f(x) + B(x) u$$

where $f(\cdot)$ admits the representation

$$f(x) = A(x) x$$

Theorem 3.1 Let

$$u = k(x) = K(x) x$$

be a stabilizing feedback law that admits a differentiable Lyapunov function, $V(x)$, such that

$$DV(x) = P(x) x$$

for some matrix, $P(x)$, and

$$DV(x)(f(x) + B(x) k(x)) = x^T P(x)(A(x) + B(x) K(x)) x \leq -x^T Q(x) x$$

with

$$x^T Q(x) x > 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}$$

If

$$x^T P(x) x > 0$$

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then there exists an $E(x)$ such that

$$E(x)x = 0$$

and

$$A(x) + E(x) + B(x)K(x)$$

is pointwise Hurwitz for all $x \in \mathbb{R}^n \setminus 0$.

**Proof** Note that from Proposition 2.1, we can assume without loss of generality that

$$P(x) > 0, \quad Q(x) > 0, \quad \forall x \in \mathbb{R}^n \setminus 0$$

Define

$$A_{cl}(x) = A(x) + B(x)K(x)$$

Then by assumption

$$x^T (P(x)A_{cl}(x) + A_{cl}(x)P(x)) x \leq -2x^T Q(x)x$$

However, this does not imply that

$$v^T (P(x)A_{cl}(x) + A_{cl}(x)P(x)) v \leq -2v^T Q(x)v$$

which would imply that $A_{cl}(x)$ is pointwise Hurwitz.

Define

$$R(x) = x^T x I - xx^T$$

As in the proof of Proposition 2.1,

$$v^T (-R(x)) v \leq 0 \Leftrightarrow v = \alpha x$$

Then

$$v^T (-R(x)) v \leq 0$$

$$\Rightarrow$$

$$-v^T (P(x)A_{cl}(x) + A_{cl}(x)P(x)) v - v^T Q(x)v > 0$$

Via the S-procedure, there exists a $\tau(x)$ such that

$$P(x)A_{cl}(x) + A_{cl}(x)P(x) - \tau(x)R(x) \leq -Q(x)$$

Now define

$$E(x) = \frac{\tau(x)}{2} P^{-1}(x) R(x)$$

By construction,

$$E(x)x = 0$$

Furthermore,

$$P(x)(A_{cl}(x) + E(x)) + (A_{cl}(x) + E(x))^TP(x) \leq -Q(x)$$

which implies that

$$A_{cl}(x) + E(x) = A(x) + E(x) + B(x)K(x)$$

is Hurwitz.

There are two key assumptions in Theorem 3.1. The first is that various functions can be expressed in a quasi-linear form, in particular

$$DV(x) = P(x)x$$

$$k(x) = K(x)x$$

This representation requires very mild regularity properties at the origin.

The main assumption required by Theorem 3.1 is that

$$x^T P(x)x = x^T DV(x) > 0$$

This assumption, also used in [6], can be given the interpretation that the level sets of $V(x)$ satisfy the star-convexity property. Let

$$S_\lambda = \{x : V(x) \leq \lambda\}$$

Then

$$x \in S_\lambda \Rightarrow \alpha x \in S_\lambda, \quad \forall \alpha \in [0, 1]$$

In other words,

$$V(\alpha x) \leq V(x), \quad \forall \alpha \in [0, 1]$$

4 **SDRE Stabilizability**

The previous section showed that for a broad class of stabilizable systems, one can always find a state-dependent feedback gain that renders the
"dynamics" matrix of a quasi-linear closed-loop system—given the correct representation.

We now show that under the same assumptions as Theorem 3.1, a stabilizing state-dependent feedback gain can be found by solving a Riccati equation.

**Theorem 4.1** Under the hypotheses of Theorem 3.1, there exist matrices $E(x)$ and $\bar{Q}(x)$, and positive scalar $\gamma(x)$, such that

$$E(x)x = 0$$

$$\bar{Q}(x) > 0, \quad \forall x \in \mathbb{R}^n \setminus 0$$

and (suppressing $x$ dependence)

$$P(A + E) + (A + E)^TP + 2\bar{Q} - \gamma PBB^TP = 0$$

**Proof** By assumption, we have that

$$\inf_{u} x^T(P(x)A(x) + A^T(x)P(x) + 2\bar{Q}(x))x + 2x^TP(x)B(x)u \leq 0$$

It is easy to show that the search over $u$ can be limited to $u$ of the form

$$u = -\gamma(x)B^T(x)P(x)x$$

Therefore,

$$\inf_{\gamma \geq 0} x^T(P(x)A(x) + A^T(x)P(x) + 2\bar{Q}(x))x - \gamma 2P(x)B(x)B^T(x)P(x)x \leq 0$$

Let $\gamma(x)$ denote the minimal $\gamma$ that satisfies the above inequality. Then

$$x^T(P(x)A(x) + A^T(x)P(x) + 2\bar{Q}(x))x - \gamma(x)P(x)B(x)B^T(x)P(x)x \leq 0$$

Following the same approach as in the proof of Theorem 3.1 leads to the construction of the desired matrices $E(x)$ and $\bar{Q}(x)$.

**References**


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