Existence of SDRE Stabilizing Feedback*

Jeff S. Shamma
Department of Mechanical and Aerospace Engineering
University of California Los Angeles
Box 951597
Los Angeles, CA 90095
shamma@seas.ucla.edu

James R. Cloutier
Eglin Air Force Base
AFRL/MNGN
101 W. Eglin Blvd.
Eglin AFB, FL 32542–6810
cloutiej@eglin.af.mil

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Abstract

The state-dependent Riccati equation (SDRE) approach to nonlinear system stabilization relies on representing a nonlinear system's dynamics in a manner to resemble linear dynamics, but with state-dependent coefficient matrices that can then be inserted into state-dependent Riccati equations to generate a feedback law. Although stability of the resulting closed loop system need not be guaranteed *a priori*, simulation studies have shown that the method can often lead to suitable control laws. In this paper, we consider the non-uniqueness of state-dependent representations. In particular, we show that if there exists any stabilizing feedback leading to a Lyapunov function with star-convex level sets, then there always exists a representation of the dynamics such that the SDRE approach is stabilizing. The main tool in the proof is a novel application of the S-procedure for quadratic forms.

1 Overview

SDRE stabilization refers to the use of State Dependent Riccati Equations to construct nonlinear feedback control laws for nonlinear systems [2, 3, 4]. The main idea is to represent the nonlinear system

$$\dot{x} = f(x) + B(x)u \tag{1}$$

in the form

$$\dot{x} = A(x)x + B(x)u \tag{2}$$

and to use the feedback

$$u = -R^{-1}(x)B^T(x)P(x)x$$

where $P(x)$ is obtained from the SDRE

$$P(x)A(x) + A^T(x)P(x) + Q(x)$$

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$$-P(x)B(x)R^{-1}(x)B^T(x)P(x) = 0$$

and $Q(\cdot)$ and $R(\cdot)$ are design parameters that satisfy the pointwise positive definiteness condition

$$Q(x) > 0 \quad R(x) > 0$$

The resulting closed loop dynamics have a quasi-linear structure

$$\dot{x} = (A(x) - R^{-1}(x)B(x)B^T(x)P(x))x$$

where the "dynamics" matrix satisfies the pointwise Hurwitz condition

$$\mathrm{Re}\lambda_i\,(\,A(x) - R^{-1}(x)B(x)B^T(x)P(x)\,) < 0$$

Although this condition is not sufficient to assure stability of the closed-loop dynamics, simulation studies have shown that this procedure is capable of deriving effective control laws.

As mentioned earlier, the matrices $Q(\cdot)$ and $R(\cdot)$ are design parameters in the SDRE approach. Another non-obvious design parameter is the choice of the representation of the original dynamics (1) in the quasi-linear form (2). It is easy to see that

$$\dot{x} = (A(x) + E(x))x + B(x)u$$

is also a representation for any matrix that satisfies

$$E(x)x = 0$$

The following is a simple illustrative example.

Example 1.1 *Consider the nonlinear plant*

$$\begin{aligned} \dot{x}_1 &= \sin(x_1) + x_2 \\ \dot{x}_2 &= x_1 x_2 + u \end{aligned} \tag{3}$$

Two equivalent representations are

$$\dot{x} = A(x)x + Bu = \begin{bmatrix} \sin(x_1)/x_1 & 1 \\ x_2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

and

$$\dot{x} = \begin{bmatrix} \sin(x_1)/x_1 & 1 \\ 0 & x_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The issue of non-uniqueness can play a major role, even affecting the controllability of the resulting parameterized pair $(A(x), B(x))$. Note that the presence or lack of controllability of this pair need not have any implication on the controllability of the original dynamics. These issues have been considered in [5].

References [4, 6] also considered the implications of non-uniqueness of the representation (2). Reference [4] derives a necessary condition on $f(x)$ and $B(x)$ for the existence of any feedback gain matrix, $G(x)$, that results in

$$A(x) + B(x)G(x)$$

being pointwise Hurwitz. Reference [6] shows that a class of nonlinear optimal control problems has the following property. The optimal feedback law

$$u = g(x)$$

can be written as

$$u = -R^{-1}(x)B^T(x)P(x)x$$

where $P(x)$ is the positive definite solution to the Riccati equation

$$\begin{aligned} & P(x)(A(x) + E(x)) + (A(x) + E(x))^T P(x) + Q(x) \\ - \quad & P(x)B(x)R^{-1}(x)B^T(x)P(x) = 0 \end{aligned}$$

for some $E(x)$ that satisfies

$$E(x)x = 0$$

In other words, there will *exist* a representation such that the SDRE feedback produces the optimal feedback law.

The following presentation follows the work of [6]. The main results are:

- If there exists any stabilizing feedback law, $u = g(x) = K(x)x$, that admits a Lyapunov function with “star-convex” level sets, then there exists a representation such that

  $$A(x) + E(x) + B(x)K(x)$$

  is pointwise Hurwitz, where $E(x)x = 0$.

- Under the same assumptions, there exists a representation, $A(x) + E(x)$, and pointwise positive definite matrices, $Q(x)$ and $R(x)$, such that the SDRE approach is stabilizing.

Loosely speaking, the implications are that if a system is stabilizable (under an appropriate Lyapunov function), then it is stabilizable via the SDRE method. The main tool in the proof is a novel application of the S-procedure for quadratic forms.

Unfortunately, these results do not suggest how to derive an appropriate representation.

Notation

For $x \in \mathcal{R}^n$, define

$$|x| = (x^T x)^{1/2}$$

For a differentiable function $V(x)$ on $\mathcal{R}^n$, $DV(x)$ denotes the matrix of partial derivatives.

2 Preliminaries

We will need the S-procedure for quadratic forms [1]. Suppose that two symmetric $n \times n$ matrices, M_0 and M_1, satisfy the following:

- For $x \in \mathcal{R}^n$, $x^T M_1 x \geq 0 \Rightarrow x^T M_0 x > 0$
- There exists a $y \in \mathcal{R}^n$ such that

$$y^T M_1 y > 0$$

Then there exists a $\tau \geq 0$ such that

$$M_0 - \tau M_1 > 0$$

The following is a consequence of the S-procedure.

Proposition 2.1 *Suppose that the $n \times n$ matrix, $P(x)$, satisfies*

$$x^T P(x) x > 0, \quad \forall x \in \mathcal{R}^n \backslash 0$$

Then there exists an $n \times n$ matrix $\tilde{P}(x)$ such that

$$P(x)x = \tilde{P}(x)x, \quad \forall x \in \mathcal{R}^n$$

and

$$v^T \tilde{P}(x) v > 0, \quad \forall x, v \in \mathcal{R}^n \backslash 0$$

Proof Define

$$M_1(x) = x^T x I - x x^T$$

Note that

$$M_1(x)x = 0$$

Furthermore, for all v

$$\begin{aligned} v^T M_1(x) v &= x^T x v^T v - v^T x x^T v \\ &= |x|^2 |v|^2 - (v^T x)^2 \\ &\geq 0 \end{aligned}$$

Therefore, if for some v

$$v^T M_1(x) v = 0$$

then necessarily $v = \alpha x$ for some $\alpha \in \mathcal{R}$. An alternative phrasing is that

$$v^T(-M_1(x))v \geq 0 \Rightarrow v^T P(x) v > 0$$

Via the S-procedure, for any $x \in \mathcal{R}^n \backslash 0$,

$$\tilde{P}(x) = P(x) + \tau(x) M_1(x) > 0$$

for some $\tau(x) \geq 0$. Furthermore by construction,

$$\tilde{P}(x)x = P(x)x$$

as desired. ∎

3 Stabilization via Representation

We now consider the stabilization of the nonlinear system

$$\dot{x} = f(x) + B(x)u$$

where $f(\cdot)$ admits the representation

$$f(x) = A(x)x$$

Theorem 3.1 *Let*

$$u = k(x) = K(x)x$$

be a stabilizing feedback law that admits a differentiable Lyapunov function, $V(x)$, such that

$$DV(x) = P(x)x$$

for some matrix, $P(x)$, and

$$\begin{aligned} & DV(x)(f(x) + B(x)k(x)) \\ = \quad & x^T P(x)(A(x) + B(x)K(x))x \\ \leq \quad & -x^T Q(x)x \end{aligned}$$

with

$$x^T Q(x)x > 0, \quad \forall x \in \mathcal{R}^n \backslash 0$$

If

$$x^T P(x)x > 0$$

then there exists an $E(x)$ such that

$$E(x)x = 0$$

and

$$A(x) + E(x) + B(x)K(x)$$

is pointwise Hurwitz for all $x \in \mathcal{R}^n \backslash 0$.

Proof Note that from Proposition 2.1, we can assume without loss of generality that

$$P(x) > 0 \quad Q(x) > 0, \quad \forall x \in \mathcal{R}^n \backslash 0$$

Define

$$A_{\text{cl}}(x) = A(x) + B(x)K(x)$$

Then by assumption

$$x^T \left(P(x)A_{\text{cl}}(x) + A_{\text{cl}}^T(x)P(x) \right) x \leq -2x^T Q(x)x$$

However, this does not imply that

$$v^T \left(P(x)A_{\text{cl}}(x) + A_{\text{cl}}^T(x)P(x) \right) v \leq -2v^T Q(x)v$$

which would imply that $A_{\mathrm{cl}}(x)$ is pointwise Hurwitz.

Define

$$R(x) = x^T x I - x x^T$$

As in the proof of Proposition 2.1,

$$v^T(-R(x))v \geq 0 \Leftrightarrow v = \alpha x$$

Then

$$v^T(-R(x))v \geq 0$$

$$\Rightarrow$$

$$-v^T\left(\, P(x)A_{\mathrm{cl}}(x) + A_{\mathrm{cl}}^T(x)P(x)\,\right)v - 2v^T Q(x) v > 0$$

Via the S-procedure, there exists a $\tau(x)$ such that

$$P(x)A_{\mathrm{cl}}(x) + A_{\mathrm{cl}}^T(x)P(x) - \tau(x)R(x) \leq -2Q(x)$$

Now define

$$E(x) = -\frac{\tau(x)}{2} P^{-1}(x)R(x)$$

By construction,

$$E(x)x = 0$$

Furthermore,

$$\begin{aligned} &P(x)(A_{\mathrm{cl}}(x) + E(x)) + (A_{\mathrm{cl}}(x) + E(x))^T P(x) \\ &\quad \leq -2Q(x) \end{aligned}$$

which implies that

$$A_{\mathrm{cl}}(x) + E(x) = A(x) + E(x) + B(x)K(x)$$

is Hurwitz. ∎

There are two key assumptions in Theorem 3.1. The first is that various functions can be expressed in a quasi-linear form, in particular

$$DV(x) = P(x)x$$

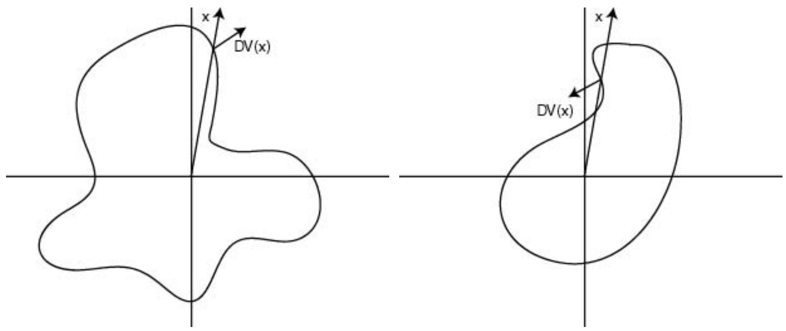

Figure 1: Level sets with and without star convexity

$$k(x) = K(x)x$$

This representation requires mild regularity properties at the origin.

The main assumption required by Theorem 3.1 is that

$$x^T P(x)x = x^T DV(x) > 0$$

This assumption, also used in [6], can be given the interpretation that the level sets of $V(x)$ satisfy the star-convexity property. Let

$$S_\lambda = \{x : V(x) \leq \lambda\}$$

Then

$$x \in S_\lambda \Rightarrow \alpha x \in S_\lambda, \quad \forall \alpha \in [0, 1]$$

In other words,

$$V(\alpha x) \leq V(x), \quad \forall \alpha \in [0, 1]$$

See Figure 1 for an illustration.

4 Example

Consider the second order system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2 \\ \dot{x}_2 &= u \end{aligned} \tag{4}$$

or alternatively in state-dependent form

$$\dot{x} = \begin{pmatrix} x_1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{5}$$

This system is easily stabilized by backstepping or feedback linearization [7].

Taking the backstepping approach leads to the change of variables

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 + x_1 + x_1^2 \end{aligned}$$

and dynamics

$$\begin{aligned} \dot{z}_1 &= -z_1 + z_2 \\ \dot{z}_2 &= (1 + 2z_1)(-z_1 + z_2) + u \end{aligned} \tag{6}$$

which suggest the stabilizing feedback

$$u = -(1 + 2z_1)(-z_1 + z_2) - z_2 \tag{7}$$

and stabilized dynamics

$$\begin{aligned} \dot{z}_1 &= -z_1 + z_2 \\ \dot{z}_2 &= -z_2 \end{aligned} \tag{8}$$

Lack of Star-Convexity

A Lyapunov function for the stabilized dynamics (8) is

$$V(z) = z_1^2 + z_2^2$$

This also provides a Lyapunov function for the original coordinates (4):

$$\tilde{V}(x) = x_1^2 + (x_2 + x_1 + x_1^2)^2$$

Figure 2 illustrates the level sets of $\tilde{V}(x)$ which do not satisfy the star-convexity requirement. This can be seen from the level set boundary twice intersecting the ray from the origin.

Pointwise Hurwitz Representation

We will illustrate the construction in Theorem 3.1 in the transformed coordinates (6).

Two different state-dependent representations for the transformed coordinates are

$$\dot{z} = \begin{pmatrix} -1 & -1 \\ -(1 + 2z_1) & (1 + 2z_1) \end{pmatrix} z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

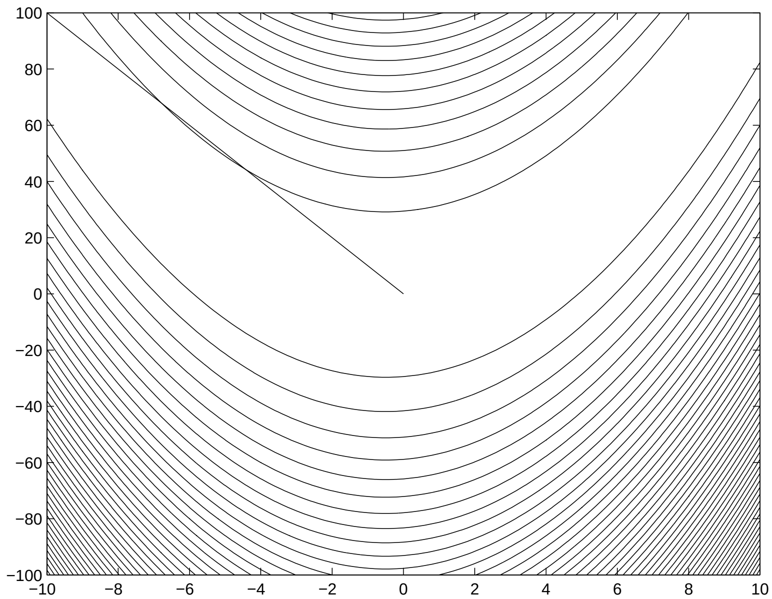

Figure 2: Level sets without star convexity

or

$$\dot{z} = \begin{pmatrix} -1 & 1 \\ -(1+2z_1-2z_2) & 1 \end{pmatrix} z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

The stabilizing feedback (7) can be represented as

$$u = \left((1+2x_1) \quad -(1+2x_1)-1 \right) z$$

which, when substituted into the state-dependent dynamics, leads to closed-loop dynamics of either

$$\dot{z} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} z$$

or

$$\dot{z} = \begin{pmatrix} -1 & 1 \\ 2z_2 & -(1+2z_1) \end{pmatrix} z$$

The first representation clearly has a pointwise Hurwitz—indeed state-independent—dynamics matrix. The latter does not have a pointwise Hurwitz dynamics matrix. The condition

$$2z_2 = 1 + 2z_1$$

implies linear dependence of the columns, and hence a zero eigenvalue.

Define

$$A_{\text{cl}}(z) = \begin{pmatrix} -1 & 1 \\ 2z_2 & -(1+z_1) \end{pmatrix}$$

According to Theorem 3.1, the existence of a Lyapunov function $V(z) = z^T P(z) z$ with $P(z) > 0$ implies the existence of a $\tau(z)$ such that

$$P(z)A_{\text{cl}}(z) + A_{\text{cl}}(z)^T P(z) - \tau(z)R(z) < 0$$

where

$$R(z) = (\, z^T z I - z z^T \,)$$

Then the matrix

$$A_{\text{cl}}(z) - \frac{\tau(z)}{2} P^{-1}(z) R(z)$$

is pointwise Hurwitz.

Applying this argument with $V(z) = z^T z$ leads to the condition

$$A_{\text{cl}}(z) + A_{\text{cl}}^T(z) - \tau(z)R(z) < 0$$

or more explicitly

$$\begin{pmatrix} 2 + \tau(z) z_2^2 & -(1 + 2z_2 + \tau(z) z_1 z_2) \\ -(1 + 2z_2 + \tau(z) z_1 z_2) & 2 + 4z_2 + \tau(z) z_1^2 \end{pmatrix} > 0$$

Positive definiteness imposes the following constraints on the diagonal terms

$$2 + \tau(z) z_2^2 > 0$$

$$2 + 4z_1 + \tau(z) z_1^2 > 0$$

Similarly, the constraint on the determinant can be written as

$$2\tau(z)\,(\,(z_1 - \tfrac{1}{2} z_2)^2 + \tfrac{3}{4} z_2^2\,) > 1 + 4z_2^2 + 4z_2 - 4 - 8z_1$$

All of these constraints impose lower bounds on $\tau(z)$ and hence can be simultaneously satisfied by a suitably large $\tau(z)$.

5 SDRE Stabilizability

The previous sections showed that for a broad class of stabilizable systems, one can always find a state-dependent feedback gain that renders the "dynamics" matrix of a quasi-linear closed-loop system pointwise Hurwitz—given the correct representation.

We now show that under the same assumptions as Theorem 3.1, a stabilizing state-dependent feedback gain can be found by solving a Riccati equation.

Theorem 5.1 *Under the hypotheses of Theorem 3.1, there exist matrices $E(x)$ and $\tilde{Q}(x)$, and positive scalar $\gamma(x)$, such that*

$$E(x)x = 0$$

$$\tilde{Q}(x) > 0, \quad \forall x \in \mathcal{R}^n \backslash 0$$

and SDRE (suppressing x dependence)

$$P(A+E) + (A+E)^T P + \tilde{Q} - \gamma P B B^T P = 0$$

whose solution leads to the stabilizing feedback

$$u = -\gamma(x) B^T(x) P(x) x$$

Proof By assumption, we have that

$$\begin{aligned} &\inf_u x^T (P(x)A(x) + A^T(x)P(x) + 2Q(x))x \\ &\quad +2x^T P(x)B(x)u \leq 0 \end{aligned}$$

It is easy to show that the search over u can be limited to u of the form

$$u = -\gamma(x) B^T(x) P(x) x$$

Therefore,

$$\begin{aligned} &\inf_{\gamma > 0} x^T (P(x)A(x) + A^T(x)P(x) + 2Q(x) \\ &\quad -\gamma 2 P(x)B(x)B^T(x)P(x))x \leq 0 \end{aligned}$$

Let $\underline{\gamma}(x)$ denote the minimal $\gamma > 0$ that satisfies the above inequality. Then

$$\begin{aligned} &x^T (P(x)A(x) + A^T(x)P(x) + 2Q(x) \\ &\quad -\underline{\gamma}(x)P(x)B(x)B^T(x)P(x))x \leq 0 \end{aligned}$$

which implies that the feedback

$$u = -\gamma(x) B^T(x) P(x) x$$

is stabilizing. Following the same approach as in the proof of Theorem 3.1 leads to the construction of the desired matrices $E(x)$ and $\tilde{Q}(x)$. ∎

6 Concluding Remarks

A key assumption in the results of this paper is the existence of a stabilizing feedback law for which a Lyapunov function with star-convex level sets is available. Under this condition, we have shown that there always exists a state-dependent representation of the open-loop state dynamics such that

- The closed-loop dynamics matrix in a state-dependent representation is pointwise Hurwitz.
- An alternative stabilizing feedback can be found via solving a state-dependent Riccati equation.

Unfortunately, the constructive procedure given here requires knowledge such a Lyapunov function.

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