Online Appendix

Forward-Looking Behavior in Mobile Data Consumption and Targeted Promotion Design: A Dynamic Structural Model

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A Proofs of Propositions in Section 5.1

To simplify notations, without causing confusion, below we suppress subscripts $i$ and $t$. Also notice that throughout all the proofs below, wherever *strict* monotonicity or concavity applies, we explicitly stress it; without explicit stress of strictness, we mean weak monotonicity/concavity.

**Proof of Proposition 1.** Myopic users determine their daily usage by maximizing the period utility only, which is defined in (1). The optimal daily usage $a^*$ can thus be derived as

$$a^* = \begin{cases} 
\mu + \xi - \eta p (> q) & \text{if } 0 < q < \mu + \xi \leq \eta p \\
q & \text{if } \mu + \xi \leq q \leq \mu + \xi \\
\max \{\mu + \xi, 0\} (< q) & \text{if } q > \mu + \xi 
\end{cases} \quad (A.1)$$

In any day before the day when the data plan quota is fully expended, $a^* = \max \{\mu + \xi, 0\} (< q)$, which is obviously independent of $q$. Q.E.D.

In order to prove Proposition 2, we first prove two key lemmas with regard to the properties of the expected value function $\bar{V}(q, d)$ as defined in (5).

**Lemma A.1.** *For the last period, the expected value function $\bar{V}(q, d = 1)$ is continuous, increasing, differentiable, and strictly concave in the remaining data plan quota $q$.***

**Proof.** Recall that in the last period,

$$V(q, d = 1, \xi) = \max_{a \geq 0} \left[ (\mu + \xi) a - \frac{1}{2} a^2 - \eta p \max \{a - q, 0\} \right] \quad (A.2)$$
We can thus explicitly solve the value function as

\[
V(q, d = 1, \xi) = \begin{cases} 
\frac{1}{2} (\mu + \xi - \eta p)^2 + \eta p q & \text{if } 0 \leq q < \mu + \xi - \eta p \\
(\mu + \xi) q - \frac{1}{2} q^2 & \text{if } \mu + \xi - \eta p \leq q \leq \mu + \xi \\
\frac{1}{2} (\max \{\mu + \xi, 0\})^2 & \text{if } q > \mu + \xi,
\end{cases}
\]

(A.3)

where \(q \geq 0\). It is easy to show that \(V(q, d = 1, \xi)\) is continuous, increasing, differentiable, and concave in \(q\) given any \(\xi\). The continuity can be easily verified by checking the function value at each endpoint. (Notice that because \(q \geq 0\), if \(\mu + \xi - \eta p < \mu + \xi < 0\), only the third segment applies and (A.3) reduces to a constant so that \(V(q, d = 1, \xi) \equiv 0\) for all \(q \geq 0\); if \(\mu + \xi - \eta p < 0 < \mu + \xi\), (A.3) reduces to two segments.) The monotonicity is immediate because the piecewise function is continuous and piecewise increasing in \(q\). \(V(q, d = 1, \xi)\) is differentiable because the left and right derivatives are equal at each endpoint: \(\frac{\partial}{\partial q} V_{q \rightarrow (\mu + \xi - \eta p)^+} (q, d = 1, \xi) = \frac{\partial}{\partial q} V_{q \rightarrow (\mu + \xi - \eta p)^-} (q, d = 1, \xi) = \eta p\), and \(\frac{\partial}{\partial q} V_{q \rightarrow (\mu + \xi)^+} (q, d = 1, \xi) = \frac{\partial}{\partial q} V_{q \rightarrow (\mu + \xi)^-} (q, d = 1, \xi) = 0\). \(V(q, d = 1, \xi)\) is concave in \(q\) because it is differentiable and piecewise concave in \(q\).

Given that \(V(q, d = 1, \xi)\) is continuous, increasing, and differentiable in \(q\) for any \(\xi\), it is immediate that the expected value function \(\bar{V}(q, d) = E_\xi V(q, d, \xi)\), as an integral over all \(\xi\), is also continuous, increasing, and differentiable in \(q\).

To show that \(\bar{V}(q, d)\) is strictly concave in \(q\), note that because \(V(q, d = 1, \xi)\) is concave in \(q\), by the definition of concavity, for any \(q_1, q_2 > 0\) and \(\lambda \in (0, 1)\), we have

\[
V((1 - \lambda) q_1 + \lambda q_2, d = 1, \xi) \geq (1 - \lambda) V(q_1, d = 1, \xi) + \lambda V(q_2, d = 1, \xi)
\]

(A.4)

for any \(\xi\). Because \(V(q, d = 1, \xi)\) is strictly concave in the second segment in (A.3), strict inequality holds in (A.4) when \(\xi \in (-\mu + q_1, -\mu + q_1 + \eta p) \cup (-\mu + q_2, -\mu + q_2 + \eta p)\). Recall that \(\xi\) is a random variable with a continuous support over the entire real field. Therefore, when taking expectation over \(\xi\) on both sides of (A.4), we have

\[
E_\xi V((1 - \lambda) q_1 + \lambda q_2, d = 1, \xi) > (1 - \lambda) E_\xi V(q_1, d = 1, \xi) + \lambda E_\xi V(q_2, d = 1, \xi),
\]

(A.5)

which shows \(\bar{V}(q, d = 1)\) is strictly concave in \(q\). Q.E.D.
Lemma A.2. If the expected value function for the next period, $\bar{V}(q', d - 1)$, is continuous, increasing, differentiable, and strictly concave in $q'$, then the expected value function for the current period, $\bar{V}(q, d)$, is also continuous, increasing, differentiable, and strictly concave in $q$.

Proof. We first show that given any $\xi$, the value function $V(q, d, \xi)$ is continuous, increasing, differentiable, and concave in $q$ if $\bar{V}(q', d - 1)$ is continuous, increasing, differentiable, and strictly concave in $q'$. Substituting (1) and (3) into (4), we can rewrite the current-period value function as

$$V(q, d, \xi) = \max_{a \geq 0} \left[ (\mu + \xi) a - \frac{1}{2} a^2 - \eta \beta [a - q] + \beta \bar{V}(q - a, d - 1) \right], \quad (A.6)$$

where $[\cdot]^+$ stands for $\max\{\cdot, 0\}$. To simplify notation, we use $\bar{V}(q, d)$ to represent $\frac{\partial}{\partial q} \bar{V}(q, d)$ for the rest of this proof.

Let $\tilde{a}$ be the solution to the first order condition (with respect to $a$) when $a < q$, that is,

$$\mu + \xi - \tilde{a} - \beta \bar{V}(q - \tilde{a}, d - 1) = 0 \quad (A.7)$$

Therefore, $\tilde{a} < q$ if and only if $\mu + \xi - q - \beta \bar{V}(0, d - 1) < 0$. When $a > q$, the first order condition yields

$$\mu + \xi - a^* - \eta \beta \bar{V}(0, d - 1) = 0. \quad (A.8)$$

$a^* > q$ if and only if $\mu + \xi - q - \eta \beta \bar{V}(0, d - 1) > 0$. Notice that $\beta \bar{V}(0, d - 1) < \eta \beta$ given $\beta < 1$. Therefore, we can summarize the optimal usage in the current period as

$$a^*(q, d, \xi) = \begin{cases} 
\mu + \xi - \eta \beta (q) & \text{if } 0 \leq q < \mu + \xi - \eta \beta \\
q & \text{if } \mu + \xi - \eta \beta \leq q \leq \mu + \xi - \beta \bar{V}(0, d - 1) \\
\max\{\tilde{a}, 0\} & \text{if } q > \mu + \xi - \beta \bar{V}(0, d - 1)
\end{cases} \quad (A.9)$$

Again, because $q \geq 0$, if $\mu + \xi - \beta \bar{V}(0, d - 1) < 0$ or $\mu + \xi - \eta \beta < 0$, (A.9) reduces to one or two
segments only. Accordingly, the current-period value function can be written as

\[
V(q, d, \xi) = \begin{cases} 
\frac{1}{2} (\mu + \xi - \eta \bar{p})^2 + \eta \bar{p}q + \beta \bar{V} (0, d - 1) & \text{if } 0 \leq q < \mu + \xi - \eta \bar{p} \\
(\mu + \xi) q - \frac{1}{2} q^2 + \beta \bar{V} (0, d - 1) & \text{if } \mu + \xi - \eta \bar{p} \leq q \leq \mu + \xi - \beta \bar{V} q (0, d - 1) \\
F(q; \xi) & \text{if } q > \mu + \xi - \beta \bar{V} q (0, d - 1),
\end{cases}
\]

(A.10)

where \( F(q; \xi) \) is defined by substituting the optimal usage \( a^* = \max \{ \bar{a}, 0 \} \) (< \( q \)) into (A.6), that is,

\[
F(q; \xi) = (\mu + \xi) a^* - \frac{1}{2} a^*^2 + \beta \bar{V} (q - a^*, d - 1).
\]

(A.11)

It is easy to show that \( V(q, d, \xi) \) is continuous in \( q \) by verifying the continuity of function value at the endpoints: for example, when \( q = \mu + \xi - \beta \bar{V} q (0, d - 1) \), \( a^* = q \) so \( F(q; \xi) = (\mu + \xi) q - \frac{1}{2} q^2 + \beta \bar{V} (0, d - 1) \). To show that \( V(q, d, \xi) \) is increasing in \( q \), we just need to show \( F(q; \xi) \) is increasing in \( q \), because it is obviously true for the first two segments of (A.10). Taking derivative with respect to \( q \) on both sides of (A.11), by Envelope Theorem, we have

\[
F_q(q; \xi) = \beta \bar{V}_q (q - a^*, d - 1) \geq 0,
\]

(A.12)

because \( \bar{V} (q', d - 1) \) is increasing in \( q' \). Therefore, \( F(q; \xi) \) is increasing in \( q \); so is \( V(q, d, \xi) \).

It is easy to show that \( V(q, d, \xi) \) is differentiable in \( q \), noticing that

\[
\frac{\partial}{\partial q} V_{q \rightarrow (\mu + \xi - \beta \bar{V} (0, d - 1))} (q, d, \xi) = \beta \bar{V}_q (0, d - 1)
\]

(A.13)

\[
\frac{\partial}{\partial q} V_{q \rightarrow (\mu + \xi - \beta \bar{V} (0, d - 1))}^+ (q, d, \xi) = F_q(q; \xi) = \beta \bar{V}_q (0, d - 1),
\]

(A.14)

where (A.14) holds by (A.12) and the fact that \( a^* = q \) when \( q = \mu + \xi - \beta \bar{V} q (0, d - 1) \).

We next show that \( V(q, d, \xi) \) is concave in \( q \). It is obvious that \( V(q, d, \xi) \) is concave when \( 0 \leq q < \mu + \xi - \eta \bar{p} \) and strictly concave when \( \mu + \xi - \eta \bar{p} \leq q \leq \mu + \xi - \beta \bar{V} q (0, d - 1) \). Given that \( V(q, d, \xi) \) is differentiable in \( q \), therefore, we only need to show that \( F(q; \xi) \) is (strictly) concave in \( q \) for \( q > \mu + \xi - \beta \bar{V} q (0, d - 1) \).

We prove by the definition of concavity. Consider any \( q_1, q_2 \geq \max \{ \mu + \xi - \beta \bar{V} q (0, d - 1), 0 \} \),
let \( \hat{q}_1 = q_1 - a^* (q_1) \) and \( \hat{q}_2 = q_2 - a^* (q_2) \). In other words, we use \(^*\) to represent the remaining quota at the beginning of the next period as a result of the optimal amount of usage in the current period. Note that \( 0 \leq \hat{q}_1 \leq q_1 \) and \( 0 \leq \hat{q}_2 \leq q_2 \). Denote \( \bar{q} = \lambda q_1 + (1 - \lambda) q_2 \) and \( \tilde{q} = \lambda \hat{q}_1 + (1 - \lambda) \hat{q}_2 \) for \( \forall \lambda \in (0, 1) \). Clearly, \( 0 \leq \tilde{q} \leq \bar{q} \). In addition, define \( U \left( q, q' \right) = (\mu + \xi) \left( q - q' \right) - \frac{1}{2} (q - q')^2 \). It is easy to show that \( U \left( q, q' \right) \) is concave in \( (q, q') \) because it is a quadratic function with a negative semidefinite Hessian matrix. Hence, \( F (\hat{q}, \xi) \) from (A.11) can be rewritten as

\[
F (\hat{q}; \xi) = U (\tilde{q}, \hat{q}) + \beta \tilde{V} (\hat{q}, d - 1) \\
\geq U (\bar{q}, \hat{q}) + \beta \tilde{V} (\hat{q}, d - 1) \\
\geq \lambda U (q_1, \hat{q}_1) + (1 - \lambda) U (q_2, \hat{q}_2) + \beta \tilde{V} (\hat{q}, d - 1) \\
\geq \lambda U (q_1, \hat{q}_1) + (1 - \lambda) U (q_2, \hat{q}_2) + \beta \lambda \tilde{V} (\hat{q}_1, d - 1) + \beta (1 - \lambda) \tilde{V} (\hat{q}_2, d - 1) \\
= \lambda F (q_1; \xi) + (1 - \lambda) F (q_2; \xi) \tag{A.15}
\]

The first inequality in (A.15) holds because of the optimality of \( \hat{q} \); the second inequality holds because of the concavity of \( U \left( q, q' \right) \) in \( (q, q') \); the third (strict) inequality holds because of the strict concavity of \( \tilde{V} \left( q', d - 1 \right) \) in \( q' \). As a result, \( F (q; \xi) \) is strictly concave in \( q \) for any \( q \geq \max \left\{ \mu + \xi - \beta \tilde{V}_q \left( 0, d - 1 \right), 0 \right\} \). Therefore, \( V (q, d, \xi) \) is concave in \( q \) for any \( q \geq 0 \) and strictly concave if \( q \geq \mu + \xi - \eta p \).

Given we have shown that \( V (q, d, \xi) \) is continuous, increasing, differentiable, and concave in \( q \) for any \( \xi \), and it is strictly concave in \( q \) when \( \xi < -\mu + q + \eta p \), following the same logic as the last part of the proof of Lemma A.1, we conclude that \( \tilde{V} \left( q', d - 1 \right) = E_\xi V \left( q, d, \xi \right) \) is continuous, increasing, differentiable, and strictly concave in \( q \). Q.E.D.

**Proof of Proposition 2.** Recall the optimal usage \( a^* (q, d, \xi) \) derived in (A.9) for any \( d \geq 2 \). In any day before the day when the data plan quota is fully expended, \( a^* (q, d, \xi) = \max \{ \bar{a}, 0 \} < q \). We want to show that \( \bar{a} \), the solution to (A.7), is strictly increasing in \( q \).

Recall that \( \bar{a} (q) \) solves the first order condition

\[
\mu + \xi - \bar{a} (q) - \beta \tilde{V}_q (q - \bar{a} (q), d - 1) = 0 \tag{A.16}
\]
By Lemma A.1 and Lemma A.2, the expected value function $\bar{V}(\cdot, \cdot)$ is increasing and strictly concave in the remaining quota for any period. Therefore, for any $q' > q$,

$$
\mu + \xi - \bar{a}(q) - \beta \bar{V}_q(q' - \bar{a}(q), d - 1) > 0
$$

(A.17)

because $\bar{V}_q(q' - \bar{a}(q), d - 1) < \bar{V}_q(q - \bar{a}(q), d - 1)$ given the strict concavity of $\bar{V}(\cdot, d - 1)$. As a result, $\bar{a}(q') > \bar{a}(q)$. Therefore, $\bar{a}$ is strictly increasing in $q$, which implies $a^*(q, d, \xi) = \max\{\bar{a}, 0\}$ is strictly increasing in $q$ if $0 < a^* < q$. Q.E.D.