A constructive theoretical platform for the description of quantum space-time crystals uncovers for $N$ interacting and ring-confined rotating particles the existence of low-lying states with proper space-time crystal behavior. The construction of the corresponding many-body trial wave functions proceeds first via symmetry breaking at the mean-field level followed by symmetry restoration using projection techniques. The ensuing correlated many-body wave functions are stationary states and preserve the rotational symmetries, and at the same time they reflect the point-group symmetries of the mean-field crystals. This behavior results in the emergence of sequences of select magic angular momenta $L_m$. For angular-momenta away from the magic values, the trial functions vanish. Symmetry breaking beyond the mean-field level can be induced by superpositions of such good-$L_m$ many-body stationary states. We show that superposing a pair of adjacent magic angular momenta states leads to formation of special broken-symmetry states exhibiting quantum space-time-crystal behavior. In particular, the corresponding particle densities rotate around the ring, showing undamped and non-dispersed periodic crystalline evolution in both space and time. The experimental synthesis of such quantum space-time-crystal wave packets is predicted to be favored in the vicinity of ground-state energy crossings of the Aharonov-Bohm-type spectra accessed via an externally applied, natural or synthetic, magnetic field. These results are illustrated here for Coulomb-repelling fermionic ions and for a lump of contact-interaction attracting bosons.

DOI: 10.1103/PhysRevA.96.043610

I. INTRODUCTION

Ground-breaking experimental progress [1–11] in the field of trapped ultracold ions and neutral atoms, in particular the unprecedented control of interparticle interactions and the attainment of ultracold temperatures, offer these systems as prime resources for experimental realization of the emergent exciting concept of a quantum space-time crystal (QSTC). Inspired by the relativistic 3+1-dimensions analogy [12], the QSTC idea extends translational symmetry breaking (SB) to encompass both the spatial and time dimensions. Indeed, the original QSTC proposal [12,13] motivated an abundance of scientific discussion, commentary, and exploration [14–24].

The original QSTC was proposed in the form of crystalline spatial-particle-density arrangements [13], or other solitonic-type (charge-density-wave) formations [12] revolving around a ring-shaped ultracold trap without dispersion or damping. Although significant experimental progress has been reported toward this goal [5,6], formation of a QSTC in this experimental configuration is yet to be demonstrated. At the same time, experimental progress for a “weaker class” [25] of discrete-time crystals [20–24] limited exclusively to the time domain has been reported [26,27], employing time-periodically driven spin systems. Contributing to this state of affairs are limitations of earlier theoretical treatments of the QSTC that were discussed extensively in previous commentary [14–16,19], e.g., limiting oneself to mean-field (MF) dynamics [12] or considering solely the energetics of states with good total angular momenta which (as a matter of principle) have uniform spatial densities [13,28]. To throw further light on the nature and properties of QSTC’s, it is imperative that a formulation and implementation of appropriate many-body trial wave functions for the QSTC on a ring be advanced. The sought-after trial wave functions should explore for a finite system of $N$ particles the interplay [29] between the mean-field symmetry-broken states, which are not eigenstates of the total angular momentum $\hat{L}$, and the exact symmetry-preserving (good total-angular-momentum) states.

Here, we introduce such trial wave functions and analyze their spectra and combined spatially dispersionless and temporally undamped evolution, which are the defining characteristics of a QSTC. Contrasting with these findings, previous beyond-mean-field theoretical studies [30–34] that investigated spatial solitonic formations in finite boson systems in one dimension or on a ring have revealed drastically different behaviors, such as increasing dispersion with time accompanied by a revival at the initial position of the propagated inhomogeneous wave packet [35].

We employ a beyond-mean-field methodology of symmetry restoration via projection techniques, introduced by us previously [29,36–44] for two-dimensional semiconductor quantum dots (with and without an applied magnetic field $\mathbf{B}$). The multilevel symmetry-breaking and symmetry-restoration approach which we pursue provides a complete theoretical framework for treating symmetry-breaking aspects in finite systems, without reference to the $N \to \infty$ limit. Indeed this approach originated, and is widely employed, in nuclear physics and chemistry [29,45–53].

The paper is organized as follows. In Sec. II we introduce and illustrate the hierarchical, multilevel methodology that we use for the construction of the trial wave functions for the microscopic many-body Hamiltonian of few ultracold ring-confined interacting particles. Following a short synopsis of the method, we discuss first in Sec. II A the mean-field, broken-symmetry state, and subsequently in Sec. II B a beyond-mean-field level is outlined, entailing symmetry restoration via the use of an angular momentum projection technique. This results in many-body stationary-state
FIG. 1. Aharonov-Bohm-type, quantum-rigid-rotor energy spectrum [second term in the right-hand side of Eq. (10)] as a function of the magnetic flux through the ring, $\Phi/\Phi_0$, for the symmetry-restored (stationary) states $\Phi_2^{{\text{PROJ}}}$ for $N = 7$ fermionic ions. The remaining parameters are Wigner parameter $R_W = 1000$, ring radius $R = 200l_0$, and oscillator strength $l_0 = 50$ nm. According to Table I in Appendix A, the parameter $C_R$ in Eq. (10) was taken equal to $1.7847 \times 10^{-3}h_0$. Each curve is labeled with the corresponding magic total angular momentum $L_m$. The circles highlight several energy-crossing points most susceptible to symmetry breaking. Energies in units of $10^{-4}h_0$

Investigated in the paper, as well as explicit expressions for the conditional probability distribution and single-particle density.

II. INTERPLAY BETWEEN SYMMETRY-BROKEN AND SYMMETRY-PRESERVING STATES: GROUP-THEORETICAL FORMULATION

In connection with the QSTC, we consider the following three levels of many-body trial wave functions. (1) A Slater determinant for localized fermions (or permanent for localized bosons) on the ring. We denote this wave function by $\Psi^{{\text{SB}}}$; it corresponds to the unrestricted Hartree-Fock (UHF), or Gross-Pitaevskii (GP), mean-field step [29] that exhibits symmetry breaking of the space degrees of freedom. $\Psi^{{\text{SB}}}$ does not preserve the total angular momentum. Out of the three levels in the hierarchical scheme (see below), it is the trial wave function closest to the familiar concept of a classical Wigner crystal [54]. (2) A stationary multideterminental (multiapermanental) wave function $\Phi_2^{{\text{PROJ}}}$ characterized by a good total angular momentum $hL$, which is generated by applying a projection operator $P_L$ (see below) on $\Psi^{{\text{SB}}}$. This step goes beyond the MF approximation and restores (as required) the quantum many-body Hamiltonian symmetries in the stationary-state solutions. Unlike $\Psi^{{\text{SB}}}$, $\Phi_2^{{\text{PROJ}}}$ exhibits an azimuthally uniform single-particle density $[\text{SPD}, \rho(r,t)]$, which is also time-independent (stationary). Previously, we referred to such projected wave functions $\Phi_2^{{\text{PROJ}}}$ as quantum rotating Wigner molecules [43]. (3) Coupling between the stationary states (brought about by a perturbation which we term in the following as “the pinning agent”) results in a superposition of two projected wave functions with different angular momenta $L_1$ and $L_2$, leading to formation of a pinned Wigner molecule (PWM), i.e.,

$$\Phi_{{\text{PWM}}}(L_1,L_2;t = 0) = \alpha \Phi_2^{{\text{PROJ}}}(L_1) + \beta e^{i\phi(t = 0)} \Phi_2^{{\text{PROJ}}}(L_2),$$

where $\phi(t = 0)$ can be set to zero without loss of generality and $\alpha^2 + \beta^2 = 1$.

In the following we illustrate the case of $\alpha = \beta = 1/\sqrt{2}$ (the physics of the PWM maintains for other choices of the mixing coefficients). For selected magic (see below) $L_1$ and $L_2$, $\Phi_{{\text{PWM}}}(L_1,L_2; t = 0)$ represent a special family of quantum wave packets with broken azimuthal symmetry. Consequently their correspoding single-particle density $\rho(r,t = 0)$ are not uniform, forming instead a crystal-like particle density pattern, with $kN$, $k = 1, 2, 3, \ldots$ possible peaks for $N$ fermionic ions and $1, 2, 3, \ldots$ possible peaks for $N$ attractive bosons. When the pinning agent is lifted, the $\Phi_{{\text{PWM}}}(L_1,L_2; t)$ evolve in time undamped according to the exact many-body quantum Hamiltonian dynamics, i.e., the phase $\phi$ will vary as $\phi(t) = (E_2 - E_1)t/h$, and the associated $\rho(r,t)$ will oscillate at any given space point with a time period $T = \tau/n; \tau = 2\pi h/E_1 - E_2; E_{1(2)}$ being the energies of the stationary states $\Phi_2^{{\text{PROJ}}}$ with $i = 1, 2$, respectively. For either statistics (fermions or bosons), $n = N$ for repelling ions and $n = 1$ for attractive particles; as aforementioned, here we discuss explcitly Coulomb-repelling fermionic ions and contact-interacting attractive bosons. Such undamped and dispersionless periodic time variation is not possible for the MF Hartree-Fock (or Gross-Pitaevskii) wave packet $\Psi^{{\text{SB}}}$, because it contains all the possible angular momenta when

Section III is devoted to further elaboration on three main topics. In Sec. III A we discuss the symmetry properties of the symmetry-restored (projected) wave functions and the selection rules for their “magic” angular momenta. Section III B analyzes the properties of the initial wave packets and their associated time evolution and Sec. III C comments on the relation between the constructed trial functions (in particular the aforementioned symmetry-restored stationary states) and the wave functions obtained through exact-diagonalization [configuration-interaction (CI)] solutions of the microscopic many-body Hamiltonian.

We conclude in Sec. IV with a summary of our work, including a brief listing of recent progress achieved in developing experimental techniques for preparation and measurement of ring-confined ultracold particles. The Appendices give tables of numerical results (rotational energies for different magic angular momenta and moments of inertia) for the systems.
expanded in the complete basis set of the stationary wave functions \( \Phi_{\text{PROJ}} \). Additionally, the MF wave functions lose [55] their single-determinant (single-permanent) character under the exact time evolution.

The many-body Hamiltonian of \( N \) identical particles in a ring-type trap threaded by a constant magnetic field \( B \) is

\[
\mathcal{H} = \sum_{i=1}^{N} \left( \frac{p_i^2}{2M} + \frac{(r_i - R)^2}{2\hbar^2/\Omega} \right) + \sum_{i<j} V(r_{ij}),
\]

(2)

where \( A(r) = B \times r/2 \) is the vector potential in the symmetric gauge, \( \hbar \sqrt{\frac{\lambda}{\Omega}} \) is the magnetic length, \( \Omega = \sqrt{\omega_0^2 + \omega_e^2/4} \) where \( \omega_0 = eB/M \) is the cyclotron frequency. The phase in Eq. (3) is due to the gauge invariance of magnetic translations [57,58] and is given by \( \psi(r, R; B) = \left(sY_Y - yX_Y\right)/\sqrt{l_0^2} \), with \( l_0 = \sqrt{\hbar/(\eta B)} \) being the magnetic length. For simplicity, in the following we provide examples for only three cases: (i) that of \( N \) fully polarized fermionic ions with odd \( N \), (ii) that of \( N \) fully polarized fermionic ions with even \( N \), and (iii) that of \( N \) spinless bosons interacting via an attractive contact potential. As will be shown explicitly, case (ii) presents different characteristics compared to case (i).

In the case of ultracold ions repelling each other via the Coulomb interaction, we take the \( R = R_{eq}e^{2\pi(j-i)/N} \), \( j = 1, 2, \ldots, N \) to coincide with the equilibrium positions (forming a regular polygon) of \( N \) classical charges inside the anular confinement specified in Eq. (2). Then \( R_{eq} = R \) is given by the real solution of the cubic equation \( aw^3 + bw^2 + d = 0 \), where \( a = 1, b = -R, d = -l_0^2\hbar\Omega/4 \), with the Wigner parameter (the ratio between the characteristic interparticle repulsion and the kinetic zero-point energy of the ring-confined particle), \( R_{eq} = e^2/(l_0\hbar\Omega) \) [29], and \( R_{eq} = \sum_{j=1}^{N} l/j \sin [(j-1)\pi/N]. \) Then the corresponding MF wave function, \( \psi_{\text{SB}} \), is the determinant formed by the \( N \) orbitals \( u(r, R) \).

In the case of \( N \) ultracold neutral bosons attracting each other with a contact interaction \( -g_\delta(r - r_j) \), the atoms are localized at the same position, and thus \( R_{eq} = R \) and \( R_{eq} = Re^{i\phi_j}, \) \( j = 1, 2, \ldots, N \). Then the MF wave function, \( \psi_{\text{SB}} \), is the product (permanent) of the orbitals \( u(r, Re^{i\phi_j}) \). The parameter corresponding to \( R_w \) is given here by \( R_w = gM/\hbar^2 \).

**B. Second level (beyond mean field): The projected, symmetry-restored stationary state**

A stationary many-body state that preserves the total angular momentum, as well as the rotational symmetry of the annular trap, can be projected out of the symmetry-broken \( \psi_{\text{SB}} \) by applying the projection operator \( \mathcal{P}_L \),

\[
\mathcal{P}_L = \frac{1}{2\pi} \int_{0}^{2\pi} e^{i\gamma(L-L)}d\gamma,
\]

(4)

where \( L = \sum_{i=1}^{N} \gamma_i, i = 1, 2, \ldots, N \), and \( \hbar L \) is the total angular-momentum operator. Then the projected many-body state is given by

\[
\Phi_{\text{PROJ}} = \frac{1}{2\pi} \int_{0}^{2\pi} d\gamma \psi_{\text{SB}}(\gamma)e^{i\gamma L}.
\]

(5)

\( \mathcal{P}_L \) is analogous to the projection operators used in chemistry for molecular orbitals governed by point-group symmetries [39,59-61]. Such projection operators are constructed through a summation over the characters of the point group [39,60,61]; the phases \( e^{i\gamma L} \) are the characters of the rotational group in two dimensions [39,61] and the operator \( e^{-i\gamma L} \) is the corresponding generator of 2D rotations. Alternatively, Eq. (5) may be viewed as a linear superposition of all the (energy-degenerate) symmetry-broken states \( \psi_{\text{SB}}(\gamma) \), azimuthally rotated by \( \gamma \). Due to the rotational symmetry, the coefficients of this superposition, i.e., the phases \( e^{i\gamma L} \), can be determined \( a \text{ priori} \), without the need to diagonalize a Hamiltonian matrix.

The projected energies, associated with the stationary wave functions \( \Phi_{\text{PROJ}} \), are given by

\[
E_{\text{PROJ}}(L) = \int_{0}^{2\pi} h(\gamma)e^{i\gamma L}d\gamma \int_{0}^{2\pi} n(\gamma)e^{i\gamma L}d\gamma,
\]

(6)

where

\[
h(\gamma) = \langle \psi_{\text{SB}}(0) | \mathcal{H} | \psi_{\text{SB}}(\gamma) \rangle,
\]

(7)

and the norm overlap

\[
n(\gamma) = \langle \psi_{\text{SB}}(0) | \psi_{\text{SB}}(\gamma) \rangle
\]

(8)

enforces proper normalization of \( \Phi_{\text{PROJ}} \). Note that the original double integration reduces to a single integration over \( \gamma \) because \( \mathcal{P}_L^2 = \mathcal{P}_L \) and \( [\mathcal{P}_L, \mathcal{H}] = 0 \).

We note that the unrestricted HFC energies for the ansatz determinant (or permanent), \( \psi_{\text{SB}} \), before projection are simply given by

\[
E_{\text{UHF}} = h(0)/n(0).
\]

(9)
We have carried out numerical calculations to determine the rotational spectrum of the $\Phi_{\text{PROJ}}^{\gamma}$. For the calculation of $h(\gamma)$ and $n(\gamma)$, we use the rules for determinants composed of nonorthogonal orbitals; see, e.g., Ref. [62]. Similar rules apply for permanents. The numerical calculations are facilitated by the fact that the one-body and two-body matrix elements between the orbitals $u(\mathbf{r}, \mathbf{R}_j)$ have closed analytic expressions [63–65].

In all three cases [(i) odd number of repelling fermions, (ii) even number of repelling fermions, and (iii) attractive bosons], and for all values of $N \leq 10$, large localization parameters $R_W \geq 200$ and $R_0 \geq 50$, and large ratios $R/l_0 \geq 40$ that we studied, we found that indeed the numerically calculated energies of the $\Phi_{\text{PROJ}}^{\gamma}$'s according to Eq. (6) (see, e.g., Tables I, II, and III in Appendix A) can be well fitted by that of an Aharonov-Bohm-type spectrum associated with a quantum many-body rigid rotor (see also [13,35]), i.e.,

$$E_{\text{PROJ}}(L) \approx V_{\text{int}} + C_R(L - N\Phi/\Phi_0)^2.$$  \hspace{1cm} (10)

$V_{\text{int}}$ approximates the ground-state energy of the few-particle system and takes different values for different many-body wave functions.

The numerically determined coefficient $C_R$ is essentially a constant (see below). $\Phi = \pi R^2_{\text{eq}} R$ is the magnetic flux through the ring and $\Phi_0 = h/\eta$ is the magnetic flux quantum. The values of the angular momenta $L$ are not arbitrary. Because of the crystalline symmetries, as well as the symmetric or antisymmetric behavior under particle exchange, they are given by proper sequences of magic angular momenta $L_m$ (see Sec. III A below for further discussion). In particular, all values of angular momenta are allowed for the case of attractive bosons, i.e., $L_m = 0, \pm 1, \pm 2, \ldots$. For the case of fully polarized repelling fermions with $N$ odd or spinless repelling bosons with any $N$, the allowed angular momenta are restricted to the sequence $L_m = kN$, with $k = 0, \pm 1, \pm 2, \ldots$. For the case of fully polarized repelling fermions with $N$ even, the allowed angular momenta are given by a different sequence $L_m = (k + 1/2)N$, with $k = 0, \pm 1, \pm 2, \ldots$.

Due to the very large values of $R_W$ and $R_0$, the value of $C_R$ is very close to that of a classical rigid rotor, corresponding to $N$ point particles in their equilibrium configuration inside the annular confinement, i.e., $C_R \approx C_R^0 = h^2/[2I(R_0)]$, with inertia moment $I(R_0) = NMR^2_{\text{eq}}$. As mentioned examples, in Table I, Table II, and Table III of Appendix A, we list calculated energies according to Eq. (6) for an odd number $N = 7$ and an even number $N = 8$ of fermionic ions, as well as for $N = 10$ attractive bosons, respectively; $R_W = 1000$ for the repelling ions and $R_0 = 50$ for the attractive bosons. The ratio $\tilde{f} \equiv C_R/C_R^0 \approx 1$ for all $L \leq 165$ for ions and for all $L \leq 30$ for attractive bosons. As aforementioned, the rigid-rotor-type spectrum in Eq. (10) was explored earlier in the QSTC literature [13,35]; however, by itself it does not lead to the derivation of appropriate QSTC wave functions. The demonstrated agreement between the microscopically calculated rotational part of the spectrum [Eq. (6)] and the analytic second term in Eq. (10) expected for a QSTC [13,35] validates the expressions $\Phi_{\text{PROJ}}^{\gamma}$ introduced in Eq. (5) as proper trial wave functions for the QSTC.

![FIG. 2. (a) CPD of the symmetry-restored stationary (beyond MF) state $\Phi_{\text{PROJ}}^{\gamma}$ for $N = 5$ fermionic ions along the perimeter of the ring (at a radius $R_0$). The arrow at $\theta_0 = 1.1\pi$ denotes the fixed point $r_0 = R_0e^{i\theta_0} = R_0e^{i1.1\pi}$. Note the $2\pi/5$ angle between the nearest-neighbor humps and between the arrow and the two adjacent humps. Other parameters are magic angular momentum $L = L_m = N$, $R_W = 200$, $R = 40l_0$, $l_0 = 50$ nm, and $\Phi/\Phi_0 = 0.8$. There is no hump at the fixed point. CPD in units of $10^{-2}/(2\pi^{2} \lambda^{2})$. (b) SPD of the original MF state $\Psi^{\text{MF}}(\gamma = 0.1\pi)$ (a determinant) for $N = 5$ fermionic ions, exhibiting explicit symmetry breaking. Other parameters are $R = 40l_0$, and $l_0 = 50$ nm. SPD in units of $10^{-1}/(2\pi^{2} \lambda^{2})$. In contrast to the symmetry-broken MF SPD in (b), the SPD of the symmetry-restored, beyond-mean-field $\Phi_{\text{PROJ}}^{\gamma}$ is azimuthally uniform; see black dashed line in Fig. 3(a). Azimuthal angle $\theta$ in units of $\pi$.](image-url)
As a specific example of the lowering of the ground-state energy in our scheme, we report that the energy for the unrestricted ansatz determinant [see Eq. (9)] in the case of \( N = 8 \) ultracold ions on a ring of radius \( R = 200 \lambda_0 \) with \( R_W = 1000 \) (case described in Table II of Appendix A) is \( E_{\text{URB}} = 118.1771 \hbar \omega_0 \), while the corresponding restored-symmetry ground state has indeed a lower energy \( V_{\text{int}} = 117.9271 \hbar \omega_0 \). This lowering of the total energy is immense compared to the quantum of the rotational motion \( C_T^2 = 1.5614 \times 10^{-6} \hbar \omega_0 \) (see caption of Table II in Appendix A).

An illustrative case of the rigid-rotor rotational spectra encoded in the second term in Eq. (10) is displayed in Fig. 1. A main feature of these spectra are the crossing points (several of them encircled) between pairs of curves with different \( L_m \)'s. The crossings define special magnetic-field values, \( \Phi/\Phi_0 = (L_1 + L_2)/(2N) \), in the neighborhood of which the system is particularly susceptible to symmetry breaking via the intermixing of two angular momenta and the ensuing generation of the PWM wave packets [see Eq. (1)].

Because the symmetry-restored (projected) wave function \( \Phi_{\text{PROJ}}^{\text{SB}} \) [Eq. (5)] preserves the group-theoretical requirements of the continuous 2D rotational group, its single-particle density is azimuthally uniform. However, the crystalline order of the original MF (symmetry-broken) wave function \( \Psi_{\text{SB}} \) is not destroyed in the symmetry-restoration step; instead, it mutates into a hidden order, which however can be revealed via the conditional probability distribution (CPD) (density-density correlation function). The CPD is given by

\[
D(\mathbf{r}, \mathbf{r}_0) = \left\langle \Phi_{\text{PROJ}}^L \left| \sum_{i \neq j} \delta(\mathbf{r}_i - \mathbf{r}) \delta(\mathbf{r}_j - \mathbf{r}_0) \right| \Phi_{\text{PROJ}}^L \rightangle.
\]

The CPD provides the probability of finding a particle in position \( \mathbf{r} \) assuming that another one is located at the fixed point \( \mathbf{r}_0 \). Substitution of the expression [Eq. (5)] that defines \( \Phi_{\text{PROJ}}^L \) yields for \( D(\mathbf{r}, \mathbf{r}_0) \) a double integral over the azimuthal angles \( \gamma_1 \) and \( \gamma_2 \); this integral expression is given in Appendix B.

Figure 2(a) displays an illustrative example of the hidden order in the symmetry-restored wave function \( \Phi_{\text{PROJ}}^L \). The CPD in Fig. 2(a) exhibits well localized features; it contrasts with the uniform horizontal black dashed lines in Figs. 3(a) and 3(b) which describe the conditional probability distribution (CPD) (density-density correlation function) of spatially inhomogeneous \( \rho(\mathbf{r}, t) \)'s of the ring trap (at a radius \( R_{\text{e}} \)).

We stress that the fixed point \( \mathbf{r}_0 \) in the CPD is arbitrary, i.e., the four peaks in the CPD in Fig. 2(a) readjust to a different choice of \( \mathbf{r}_0 \) so that the relative distance between them and the arrow remain unchanged. Figure 2(b) displays the SPD of the original state \( \Psi_{\text{SB}}(\gamma = 0.1 \pi) \) (a determinant) for \( N = 5 \) fermionic ions, exhibiting explicitly the symmetry breaking at the mean-field level.

C. Third level (beyond mean field): Periodic time evolution of the spatially inhomogeneous \( \rho(\mathbf{r}, t) \) associated with the wave packet \( \Phi_{\text{PWM}}(L_1, L_2; t = 0) \)

As aforementioned, the two-state wave packet in Eq. (1) is not an eigenstate of the total angular momentum, and thus it is not a stationary state when the pinning agent is lifted; such a pinning agent could be implemented, for example, as a distortion of the circular geometry of the trap confinement, or as a modulation of the trap potential in the azimuthal direction along the ring [6]. (A sudden variation of the magnetic field can also transform an eigenstate \( \Phi_{\text{PROJ}}^L(B_1) \) at a given \( B_1 \) value to a superposition of \( \Phi_{\text{PROJ}}^L(B_2) \) states at another \( B_2 \) value [65].) The resulting time evolution is associated with a time-dependent phase \( \phi(t) \) as discussed previously. Here we will show explicitly that \( \phi(t) \) represents an undamped rotation of spatially inhomogeneous \( \rho(\mathbf{r}, t) \)'s around the ring, so that the many-body \( \Phi_{\text{PWM}}(L_1, L_2; t) \) exhibit the desired behavior of a QSTC.

The success of the theoretical identification and experimentally implemented superposition of two appropriate many-body spin eigenstates of the Ising Hamiltonian (resulting in a “spin Schrödinger-cat” state) was key to the emulation of the “weaker class” of discrete time crystals [21,22,24,26].

As in the case of the CPD, \( \rho(\mathbf{r}, t) \) entails a double integral over the azimuthal angles \( \gamma_1 \) and \( \gamma_2 \); the lengthy expression is given in Appendix C.

Figure 3 displays the periodic time evolution of \( \rho(\mathbf{r}, t) \)'s for two illustrative \( \Phi_{\text{PWM}}(L_1, L_2; t) \) cases, one for \( N = 5 \) Coulomb repelling fermionic ions [Fig. 3(a)] with \( L_2 - L_1 = N \) and the other for \( N = 7 \) neutral bosons with \( L_2 - L_1 = 1 \) [Fig. 3(b)] interacting via an attractive contact interaction.

The \( \rho(\mathbf{r}, t) \)'s were calculated at times \( t = j \pi/4 \), where \( \tau = \)
2πℏ/|[E^{PROJ}_L(L_2)−E^{PROJ}_L(L_1)]|; the actual used j’s label the ρ(r,t) curves. The number of humps exhibited by the PWM ρ(r,t)’s in Fig. 3(a) and Fig. 3(b) is equal to that in the original MF densities, i.e., N for the repelling-fermions PWM and one for the attractive-bosons lump. The period of the PWM ρ(r,t)’s is \(T = τ/N\) for repelling ions and \(T = τ\) for attractive bosons.

Finally, Fig. 3(c) demonstrates a different state of matter, i.e., multiharmonic excitations of the QSTC exhibiting a multiple number of density humps, i.e., \(E_1 N\) and \(k (k = 2, 3, \ldots)\), corresponding to \(E^{PWM}_1(L_1,L_2;L);\)’s with \(L_2 - L_1 = k N\) for repelling fermions and with \(L_2 - L_1 = k\) for attractive bosons, respectively.

We note that the PWM broken-symmetry state introduced here to describe a QSTC has an energy intermediate between \(E_1\) and \(E_2\) because \(α^2 + β^2 = 1\) (i.e., \(E^{PWM} = α^2 E_1 + β^2 E_2\)). In particular, at the crossing point of the two parabolas (where \(E_1 = E_2 = E_{cross}\)), one has always \(E^{PWM} = E_{cross}\). This contrasts with the behavior of the energy of the noncrystalline states studied in Refs. [67–69], which lies always well above the crossing point.

### III. DISCUSSION

#### A. Symmetries of the trial wave functions, magic angular momenta, and rigidity

Despite the fact that the trial wave functions in Eq. (5) are a good approximation to the rotational-symmetry-preserving many-body eigenstates, they do embody and reflect in an optimum way the crystalline point-group symmetries (familiar from bulk crystals). Specifically, the \(C_N\) point-group symmetry of the “classical” crystal, which is accounted for through the kernel of symmetry-broken MF determinants (or permanents) \(Ψ^{SB}_1\), is reflected in the fact that the trial wave functions \(Φ^{PROJ}_L\) are identically zero except for a subset of magic angular momenta \(L_m\).

In the case of \(N\) repelling particles, the magic total angular momenta can be determined by considering the point-group symmetry operator \(R(2π/N) ≡ \exp(−i2π\frac{L}{N})\) that rotates on the ring simultaneously the localized particles by an angle \(2π/N\).

In connection to the state \(Φ^{PROJ}_L\), the operator \(R(2π/N)\) can be invoked in two different ways, namely either by applying it on the “intrinsic” part \(Ψ^{SB}_1\) or the “external” phase factor \(\exp(iγL)\) (see Chap. 4-2c of Ref. [70]). One gets in the case of fermions

\[
\hat{R}(2π/N)Φ^{PROJ}_L = (−1)^{N−1}Φ^{PROJ}_L
\]

from the first alternative and

\[
\hat{R}(2π/N)Φ^{PROJ}_L = \exp(−2π\frac{Li}{N})Φ^{PROJ}_L
\]

from the second alternative. The \((−1)^{N−1}\) factor in Eq. (13) results from the fact that the 2π/N rotation is equivalent to exchanging \(N−1\) rows in the \(Ψ^{SB}_1\) determinant. Now if \(Φ^{PROJ}_L \neq 0\), the only way that Eqs. (13) and (14) can be simultaneously true is if the condition \(\exp(2π\frac{Li}{N}) = (−1)^{N−1}\) is fulfilled. This leads to the following sequence of magic angular momenta:

\[
L_m = k N; \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots,
\]

for \(N\) odd, and

\[
L_m = \left(k + \frac{1}{2}\right) N; \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots
\]

for \(N\) even.

Because a permanent is symmetric under the interchange of two rows, the corresponding magic \(L_m\)’s for spinless bosons are given by the sequence in Eq. (15) for both odd and even numbers of localized bosons.

Regarding the numerical aspects, the fact that \(Φ^{PROJ}_L\) is zero for nonmagic \(L\) values results in the vanishing (within machine precision) of the normalization factor \(\int_0^{2π} n(y)e^{iγL}dy\) in Eq. (6). As a result only the physically meaningful energies associated with magic angular momenta are given in Table I, Table II, and Table III of Appendix A.

We stress that the properties and physics associated with magic-angular-momentum yrast states are well known in the literature of 2D quantum dots [29,71–74]. Of immediate relevance to this paper is the enhanced energy stabilization that they acquire in their neighborhood (thus characterized often as “cusp” states) in the regime of strong interactions (i.e., for large \(R_w\) or \(R_s\)). This energy stabilization can be explicitly seen in Fig. 15 of Ref. [29], where the triplet state corresponds to the fully polarized case for two electrons with magic angular momenta \(L_m = (2k + 1), k = 0, \pm 1, \pm 2, \ldots\). The fact that large energy gaps do develop between the magic-angular-momentum rotational yrast states and the other (excited) states is also well established in the QSTC literature; for the case of ultracold ions on a ring, see Refs. [13,35], and for the case of the bosonic lump, see Ref. [12].

In this paper, we consider fully polarized fermions only, that is cases when \(S = S_z = N/2\), where \(S\) is the total spin and \(S_z\) is its projection. Consideration with our methodology of the other spin values \(S < N/2\) is straightforward; it requires, however, restoration of both the total spin \(S^2\) and the total angular momentum. An explicit example for \(N = 3\) fermions is discussed in Ref. [39].

In addition to the magic angular momenta, the properties of the original crystalline structure built-in in \(Ψ^{SB}_1\) are reflected in the high degree of rigidity exhibited by the symmetry-preserving \(Φ^{PROJ}_L\). As demonstrated previously, the SPD of \(Φ^{PROJ}_L\) is uniform, but the CPD of \(Φ^{PROJ}_L\) reveals the now hidden crystalline structure of \(N\) strongly repelling particles on the ring. The rigidity of \(Φ^{PROJ}_L\) is manifested in that the CPDs have the same \(N\)-hump shape and are independent of the actual value of the magic angular momenta, as well as of the fermion or boson statistics and of whether the number \(N\) of fermions is odd or even. This rigidity is a consequence of the strong two-body interaction and cannot be found in many-body wave functions associated with weak interparticle interactions.

#### B. Initial wave packets and associated time evolution

The focus of this paper is the construction of a symmetry-preserving wave function \(Φ^{PROJ}_L\) associated with the finite-cystal symmetry-broken determinant (or permanent) \(Ψ^{SB}_1\). However, it is instructive to inquire about the reverse process, that is how to represent the symmetry-broken crystal as a
superposition in the complete basis formed by the symmetry-preserving $\Phi^{\text{PROJ}}_L$s. Indeed one can write the expansion

$$\psi_{\text{SB}} = \sum_{L} C_L \Phi^{\text{PROJ}}_L,$$

where [using Eq. (5)] the expansion coefficients are given by

$$C_L = \frac{1}{2\pi} \int_0^{2\pi} d\gamma e^{-iL \gamma} n(\gamma),$$

and the norm overlap $n(\gamma)$ was defined in Eq. (8). Of course the index $L$ runs over the appropriate sequence of magic angular momenta as discussed in Sec. III A.

Equations (17) and (18) illustrate the fact that with respect to the exact many-body (linear) Schrödinger equation the symmetry-broken-crystal wave function $\psi_{\text{SB}}$ is a wave packet and not a stationary eigenstate. This is also in general true for all SB mean-field solutions, whether they are solutions of the unrestricted Hartree-Fock equations in the case of confined electrons (e.g., in quantum dots [29]), or they are the familiar solitonic solutions of the Gross-Pitaevskii equations in one-dimensional bosonic systems. Due to the rapid experimental control, the latter cases are currently attracting a lot of attention. Indeed motivated by experiments that suggest the need to go beyond mean-field dynamics, the number of related theoretical investigations has burgeoned [30–34].

These theoretical studies investigate how an initial state approximating a solution of the nonlinear Gross-Pitaevskii equation evolves in time under the exact many-body Hamiltonian. For both the cases of dark [30,31] (a hole in matter density) and bright [32–34] solitons (an excess in matter density, like the case of the lump considered in this paper), these studies are finding a “universal” behavior of dispersion in space, decay in time, and time revival at the initial position of the soliton. This behavior can be understood by taking into consideration the expansion in Eq. (17). In fact, each energy mode in Eq. (17) will evolve in time according to its own phase $\exp(-i E_L t/\hbar)$, and the interaction between all of them results in a decay-type behavior. Moreover, the initial occupation amplitudes $C^2_L$ of the different modes are unequal, and as a result probability flows from the higher occupied modes to those with lower initial occupations, which leads to a dispersive behavior. However, because the system is finite, there exists a Poincaré period, and the system will eventually experience a revival [75,76].

For achieving a QSTC, we propose here a different initial wave packet, i.e., a two-mode one with equal weights, as specified in Eq. (1). As explicitly demonstrated through numerical calculations, such a two-mode initial wave packet preserves at all times, and without damping, the spatial and temporal periodicities expected from the classical finite crystal. We note that the consideration of two-mode Schrödinger-cat states is a key element in the theory of the discrete time crystal, where the focus is to enable a sloshing behavior between these specialized paired states by minimizing interactions to the rest of the system. In fact, our symmetry-preserving trial functions $\Phi^{\text{PROJ}}_L$ [see Eq. (5)] can be viewed as a more complex class of Schrödinger-cat states. This analogy is straightforward for the case of the mirror superposition used in Ref. [20], which can be reproduced from expression (5) as a limiting case by using only two angles $\gamma_1 = 0$ and $\gamma_2 = \pi$.

C. Relation to configuration-interaction (CI) wave functions

As mentioned previously, the symmetry-preserving trial functions $\Phi^{\text{PROJ}}_L$s are identically zero for values of $L$ different from the magic angular momenta $L_m$; see Sec. III A. Naturally the exact many-body spectrum has a plethora of additional states with good $L$, which however cannot be reached with the approach in this paper. Indeed this approach is tuned to extracting from the complete spectrum only the ground states that correspond to nonvibrating classical finite crystal arrangements. The remaining many-body states can be reached by using the CI approach, which is in principle an exact methodology when converged; the CI is often referred to as exact diagonalization (EXD). The CI approach is computationally expensive, but comparisons between the symmetry-restored trial functions and the CI wave functions have been used by us to demonstrate the numerical accuracy of the symmetry-restored wave functions, as well as to clarify their special place in the whole spectrum, namely that for particular magnetic-field ranges they can become the global ground state, as is the case with the Aharonov-Bohm spectrum in Fig. 1 of the present paper. Higher-in-energy CI solutions with different $L$ (and also with $L = L_m$) do incorporate vibrational and other types of internal excitations, and as a result a superposition of two random CI states with good $L_1$ and $L_2$ will not necessarily exhibit the crystalline single-particle-density structure of exactly $N$ humps.

Systematic comparisons between symmetry-restored states and CI wave functions have been carried out by us previously for the case of a few electrons confined in parabolic quantum dots. Although the external confining potential and particle species in parabolic QDs are different from the case of the ring traps considered here, the symmetry properties of the many-body wave functions are universal. Thus the analysis presented in our previous QD studies can be used to gain further insights to the results for the QSTC presented in Sec. II. In particular, Fig. 6 and Fig. 7 of Ref. [44] offer an explicit illustration of the fact that $N$-humped crystalline SPD structures arise only when both $L_1$ and $L_2$ coincide with magic angular momenta and the associated CI wave functions correspond to global ground states in some range of magnetic fields (see Fig. 5 and Sec. III in Ref. [44]).

In our previous studies of QDs, excellent agreement was found between the total energies of symmetry-restored trial functions and the corresponding CI energies for both the cases with or without an applied magnetic field, as testified by the many reported direct numerical comparisons. We mention here a few specific examples, i.e., Tables III and IV in Ref. [39], Table IV in Ref. [38], and Fig. 4 in Ref. [43].

Such systematic numerical comparisons between symmetry-restored and CI wave functions for the case of ring-trapped ultracold ions and neutral bosons are outside the scope of the present paper. However, they will be reported in subsequent publications [65], including the case away from the quasi-1D regime (when the dependence on the ring width becomes important).
IV. CONCLUSIONS

The discussion [17–19], motivated by the criticism [14–16,19] of the original [12,13] QSTC proposals (which were based on ground states), spurred speculations about nonequilibrium low-lying states as possible instruments for describing QSTCs. For $N$ rotating particles on a ring, and using the theory of symmetry breaking and symmetry restoration via projection techniques [29], this paper succeeded in explicitly uncovering the existence of low-lying states with QSTC behavior, by introducing beyond-MF appropriate trial many-body wave functions (see Fig. 3). Along with its conceptual and methodological significance, we propose to focus experimental attention on selected applied magnetic-field values where the Aharonov-Bohm-type spectra corresponding to different magnetic angular momenta are most susceptible to mixing (Fig. 1), resulting in rotating pinned-Wigner-molecule many-body states found here to exhibit QSTC behavior. This constructive platform fills an apparent gap in the quest for ultracold ring-confined ions or neutral-atom QSTCs.

We recall that although the original proposals for the quantum space-time crystal [12,13] suggested realization of the concept through the use of ultracold few ring-trapped particles, this is yet to be achieved experimentally. Nevertheless, for the case of ultracold ions, several publications have reported significant progress in controlling aspects of a quantum rotor on a ring. In particular, the ability to generate and control symmetry-breaking through pinning of the rotating ion crystal has been demonstrated by using up to $15^{40}$Ca$^+$ ions in a ring with a microfabricated silicon surface Paul trap [6], or $3^{40}$Ca$^+$ ions in a 2D ring-type configuration in a linear Paul trap [7]. To fully implement and control the QSTC trial functions presented in this paper, the ions need to be cooled down to near the ground states. In this respect, Ref. [6] has achieved temperatures $\sim 3$ mK (for a trap with a radius of $\sim 60\,\mu$m), while Ref. [7] reported temperatures in the nano-Kelvin range (for an effective ring radius in the 6 to 8 micrometer range). It is expected that cooling techniques and procedures will be further optimized and will be successful in the near future in producing near-ground-state temperatures, as is exemplified by a very recent publication [77].

An essential requirement, met by our theory, is that it is imperative that the proposed beyond-mean-field many-body trial wave functions (i.e., beyond the UHF or GP treatments) for predicting proper quantum space-time-crystal behavior of particles moving on a ring will be based on solutions to the interacting particles Schrödinger equation that possess good angular momenta, as well as exhibit (hidden) ordering that reflects an underlying finite crystalline symmetry. This is achieved in our theory through the first two construction stages, namely, the unrestricted Hartree-Fock solution followed by an angular momentum projection, yielding the function $\Phi_{Lm}^{\text{Proj}}$ [Eq. (5)]. It is then proposed by us that these projected and stationary many-body wave functions are susceptible to mixing, see Eq. (1), favored to occur in the vicinity of crossings of Aharonov-Bohm-type spectra of ground-state energies vs applied magnetic field (see circles in Fig. 1). This mixing results in nonstationary low-lying states that, when evolved with the many-body Hamiltonian, yield undamped and nondispersing periodic oscillations in both space and time.

ACKNOWLEDGMENTS

Work supported by the Air Force Office of Scientific Research under Award No. FA9550-15-1-0519. Calculations were carried out at the GATECH Center for Computational Materials Science.

<table>
<thead>
<tr>
<th>$L_m$</th>
<th>$N = 7$ fermions, $\Phi = 0$</th>
<th>$N = 7$ fermions, $\Phi/\Phi_0 = 3.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{\text{Proj}}^{\text{P}}(L_m)$</td>
<td>$f$</td>
<td>$E_{\text{Proj}}^{\text{P}}(L_m)$</td>
</tr>
<tr>
<td>0</td>
<td>85.6564962006</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>85.6565836512</td>
<td>1.000011</td>
</tr>
<tr>
<td>14</td>
<td>85.6568460029</td>
<td>1.000011</td>
</tr>
<tr>
<td>21</td>
<td>85.6572832557</td>
<td>1.000011</td>
</tr>
<tr>
<td>28</td>
<td>85.6578954097</td>
<td>1.000011</td>
</tr>
<tr>
<td>35</td>
<td>85.6586824648</td>
<td>1.000011</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>105</td>
<td>85.6671275766</td>
<td>1.000011</td>
</tr>
<tr>
<td>112</td>
<td>85.6788835437</td>
<td>1.000010</td>
</tr>
<tr>
<td>119</td>
<td>85.6817694118</td>
<td>1.000010</td>
</tr>
<tr>
<td>126</td>
<td>85.6848301808</td>
<td>1.000010</td>
</tr>
<tr>
<td>133</td>
<td>85.6880658508</td>
<td>1.000010</td>
</tr>
<tr>
<td>140</td>
<td>85.6914764217</td>
<td>1.000010</td>
</tr>
</tbody>
</table>
TABLE II. Rotational energy spectra according to Eq. (6) and ratio $\tilde{f} = C_R/C_R^0$ for $N = 8$ spin-polarized ultracold fermionic ions at two different magnetic fields $\Phi = 0$ and $\Phi/\Phi_0 = 3.2$. The interparticle interaction is a repelling Coulomb potential. The energies are in units of $\hbar^2/(2Ml_0^2)$. The remaining parameters are Wigner parameter $R_W = 1000$, ring radius $R = 200l_0$, and oscillator strength $l_0 = 50$ nm. As a function of $L_m$, the numerically extracted coefficient $C_R$ in Eq. (10) was determined from the ratio $C_R = [E^\text{PROJ}(L_m) - E^\text{PROJ}(L_m - N)]/[N(2L_m - 2\Phi/\Phi_0 - N)]$. Its value is practically constant and equal to $C_R^0$; see the values of the ratio $\tilde{f}$, which are very close to unity. The classical rigid-body value is $C_R^0 = 1.5614 \times 10^{-6}\hbar\omega_0$. The underlined numbers refer to the ground state for a given $\Phi/\Phi_0$.

<table>
<thead>
<tr>
<th>$L_m$</th>
<th>$N = 8$ fermions, $\Phi = 0$</th>
<th>$f$</th>
<th>$L_m$</th>
<th>$N = 8$ fermions, $\Phi/\Phi_0 = 3.2$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>117.9270981536</td>
<td>4</td>
<td>117.9278028694</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>117.9272980155</td>
<td>12</td>
<td>117.9273627314</td>
<td>1.001015</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>117.9276977394</td>
<td>20</td>
<td>117.9271224553</td>
<td>1.001852</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>117.9282973253</td>
<td>28</td>
<td>117.9270820410</td>
<td>1.011065</td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>117.9290967731</td>
<td>36</td>
<td>117.9272414889</td>
<td>0.997248</td>
<td></td>
</tr>
<tr>
<td>44</td>
<td>117.9300960828</td>
<td>44</td>
<td>117.9276007987</td>
<td>0.998782</td>
<td></td>
</tr>
<tr>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
<td></td>
</tr>
<tr>
<td>124</td>
<td>117.9510815852</td>
<td>124</td>
<td>117.9421863015</td>
<td>0.999823</td>
<td></td>
</tr>
<tr>
<td>132</td>
<td>117.9542793756</td>
<td>132</td>
<td>117.9447449029</td>
<td>0.999837</td>
<td></td>
</tr>
<tr>
<td>140</td>
<td>117.9576770279</td>
<td>140</td>
<td>117.9475017443</td>
<td>0.999850</td>
<td></td>
</tr>
<tr>
<td>148</td>
<td>117.9612745420</td>
<td>148</td>
<td>117.9505492585</td>
<td>0.999860</td>
<td></td>
</tr>
<tr>
<td>156</td>
<td>117.9650719179</td>
<td>156</td>
<td>117.9536166343</td>
<td>0.999870</td>
<td></td>
</tr>
<tr>
<td>164</td>
<td>117.9690691554</td>
<td>164</td>
<td>117.9569738719</td>
<td>0.999878</td>
<td></td>
</tr>
</tbody>
</table>

APPENDIX A: NUMERICAL CALCULATIONS OF THE MANY-BODY ROTATIONAL ENERGIES $E^\text{PROJ}(L_m)$ [EQ. (6)]

Tables I, II, and III present three illustrative examples of the rotational energy spectra $E^\text{PROJ}(L_m)$ according to numerical calculations of the many-body expression in Eq. (6) of the main text. The captions explain how the numerical $C_R$ in Eq. (10) is extracted from the computed values of $E^\text{PROJ}(L_m)$. $C_R$ is found to be very close to the classical rigid-rotor value $C_R^0 = \hbar^2/[2I(R_{eq})]$.

TABLE III. Rotational energy spectra according to Eq. (6) and ratio $\tilde{f} = C_R/C_R^0$ for $N = 10$ spinless ultracold bosons at two different magnetic fields $\Phi = 0$ and $\Phi/\Phi_0 = 2.464$. The interparticle contact interaction is attractive. The energies are in units of $\hbar^2/(2Ml_0^2)$. The remaining parameters are Wigner parameter $R_W = 50$, ring radius $R = 40l_0$, and oscillator strength $l_0 = 1 \mu$m. As a function of $L_m$, the numerically extracted coefficient $C_R$ in Eq. (10) was determined from the ratio $C_R = [E^\text{PROJ}(L_m) - E^\text{PROJ}(L_m - 1)]/(2L_m - 2\Phi/\Phi_0 - 1)$. Its value is practically constant and equal to $C_R^0$; see the values of the ratio $\tilde{f}$, which are very close to unity. The classical rigid-body value is $C_R^0 = 3.1250 \times 10^{-5}\hbar\omega_0$. The underlined numbers refer to the ground state for a given $\Phi/\Phi_0$.

| $L_m$ | $N = 10$ bosons, $\Phi = 0$ | $f$ | $L_m$ | $N = 10$ bosons, $\Phi/\Phi_0 = 2.464$ |
|------|-----------------|-----|------|-----------------|-----|
| 0    | -350.8488583900 | 0   | -350.8303108498 |
| 1    | -350.8488271326 | 1   | -350.8318195924  | 0.999995 |
| 2    | -350.8487333607 | 2   | -350.8332658201  | 0.999885 |
| 3    | -350.8485770740 | 3   | -350.8346495329  | 0.999973 |
| 4    | -350.8483582728 | 4   | -350.8359707308  | 0.999961 |
| 5    | -350.8480769568 | 5   | -350.8372294139  | 0.999947 |
| …   | …               | …   | …               | …     |
| 23   | -350.8323232897 | 23  | -350.8491956906  | 0.997529 |
| 24   | -350.8308542000 | 24  | -350.8492665957  | 0.995160 |
| 25   | -350.8293225964 | 25  | -350.8492749867  | 0.958962 |
| 26   | -350.8277284788 | 26  | -350.8492208634  | 1.006945 |
| 27   | -350.8260718473 | 27  | -350.8491042260  | 1.003333 |
| 28   | -350.8243527019 | 28  | -350.8489250745  | 1.002246 |
| 29   | -350.8225710428 | 29  | -350.8486834090  | 1.001722 |
| 30   | -350.8207268699 | 30  | -350.8483792296  | 1.001414 |
APPENDIX B: CONDITIONAL PROBABILITY DISTRIBUTION

The explicit expression for the CPDs of the symmetry-restored wave functions $\Phi^{\text{PROF}}_L$ [see Eq. (11)] is given by

$$D(r,r_0) = \frac{\int_{0}^{2\pi} d\gamma_1 \int_{0}^{2\pi} d\gamma_2 e^{i(\gamma_1 - \gamma_2)L} \sum_{k\neq m,l \neq n} \left( G^{nl}_{km}(\gamma_1, \gamma_2) \mp G^{km}_{ln}(\gamma_1, \gamma_2) \right) S^{\text{PDI}}_{lm}(\gamma_1, \gamma_2)}{2\pi \int_{0}^{2\pi} n(\gamma) e^{i\gamma L} d\gamma},$$  \hspace{1cm} (B1)

where

$$G^{nl}_{km}(\gamma_1, \gamma_2) = \frac{1}{\pi^2 L^2} \exp\left( -\frac{(r - R_k(\gamma_1))^2 + (r - R_l(\gamma_2))^2 + (r_0 - R_m(\gamma_1))^2 + (r_0 - R_n(\gamma_2))^2}{2\lambda^2} \right) \times \exp\left( i \left( x(Y_k(\gamma_1) - Y_l(\gamma_2)) + x(Y_l(\gamma_1) - Y_k(\gamma_2)) + x_0(Y_m(\gamma_1) - Y_n(\gamma_2)) + x_0(Y_n(\gamma_1) - Y_m(\gamma_2)) \right) \right),$$ \hspace{1cm} (B2)

and the $S^{\text{PDI}}_{lm}(\gamma_1, \gamma_2)$’s are two-row ($km$)-two-column ($ln$) cofactors of the determinant (minors of the permanent) constructed out of the overlaps of the localized space orbitals $u(r, R_j)$ [Eq. (3)]. The $\mp$ sign in Eq. (B1) corresponds to fermions or bosons.

APPENDIX C: SINGLE-PARTICLE DENSITY

The explicit expression for the SPDs of the broken-symmetry wave packets $\Phi^{\text{PIN}}(L_1, L_2; t)$ [see Eq. (12)] is given by

$$\rho(r; t) = \frac{\int_{0}^{2\pi} d\gamma_1 \int_{0}^{2\pi} d\gamma_2 (-a^2 e^{i(\gamma_1 - \gamma_2) L_1 + \alpha} + a^2 e^{i(\gamma_1 - \gamma_2)L_1 + \phi(\gamma_1 - \gamma_2) L_2} + a^2 e^{i(\gamma_1 - \gamma_2)L_1 + \phi(\gamma_1 - \gamma_2) L_2}) \sum_{k \neq l} \mathcal{F}_{kl}(\gamma_1, \gamma_2) S^f_{kl}(\gamma_1, \gamma_2)}{2\pi \int_{0}^{2\pi} n(\gamma) (a^2 e^{i\gamma L_1} + a^2 e^{i\gamma L_2}) d\gamma},$$ \hspace{1cm} (C1)

where

$$\mathcal{F}_{kl}(\gamma_1, \gamma_2) = \frac{1}{\pi^2 \lambda^2} \exp\left( -\frac{(r - R_k(\gamma_1))^2 + (r - R_l(\gamma_2))^2}{2\lambda^2} \right) \exp\left( -i \left( x(Y_k(\gamma_2) - X_k(\gamma_1)) + x(Y_k(\gamma_1) - Y_k(\gamma_2)) \right) \right),$$ \hspace{1cm} (C2)

and the $S^f_{kl}(\gamma_1, \gamma_2)$’s are one-row ($k$)-one-column ($l$) cofactors of the determinant (minors of the permanent) constructed out of the overlaps of the localized space orbitals $u(r, R_j)$ [Eq. (3)].
TRIAL WAVE FUNCTIONS FOR RING-TRAPPED IONS

[28] See also the Supplemental Material of Ref. [13].
[35] Wave packet dispersion was also reported in a recent study of inhomogeneous spin configurations on a ring of N bosonic ions, F. Robicheaux and K. Niffenegger, Quantum simulations of a freely rotating ring of ultracold and identical bosonic ions, Phys. Rev. A 91, 063618 (2015).


